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Dyon spectrum in generic $\mathcal{N} = 4$ supersymmetric $\mathbb{Z}_N$ orbifolds

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Abstract: We find the exact spectrum of a class of quarter BPS dyons in a generic $\mathcal{N} = 4$ supersymmetric $\mathbb{Z}_N$ orbifold of type IIA string theory on $K3 \times T^2$ or T^6. We also find the asymptotic expansion of the statistical entropy to first non-leading order in inverse power of charges and show that it agrees with the entropy of a black hole carrying same set of charges after taking into account the effect of the four derivative Gauss-Bonnet term in the effective action of the theory.

Keywords: D-branes, Black Holes in String Theory.

Contents

1. Introduction and summary

We now have a good understanding of the spectrum of 1/4 BPS states in a class of $\mathcal{N} = 4$ supersymmetric string theories which are obtained as $\mathbb{Z}_N$ orbifolds of type IIA string theory on $K3 \times T^2$ or $T^4 \times T^2$ for prime values of N [1 − 10]. In each example studied so far, the statistical entropy computed by taking the logarithm of the degeneracy of states agrees with the entropy of the corresponding black hole for large charges, not only in the leading order but also in the first non-leading order [2, 6, 9, 10]. On the black hole side this requires taking into account the effect of Gauss-Bonnet term in the low energy effective action of the theory, and use of Wald's generalized formula for the black hole entropy in the presence of higher derivative corrections [11 − 14].

In this paper we generalize this analysis to $\mathcal{N} = 4$ supersymmetric theories, obtained as $\mathbb{Z}_N$ orbifolds of type IIA string theory on $K3 \times T^2$ or $T^4 \times T^2$, for generic N which are not necessarily prime. In this process we also demonstrate the relationship between the black hole entropy and the statistical entropy in a more explicit manner by comparing the expressions for various coefficients rather than matching their final values.

Since the analysis of the paper involves a lot of technical details, we shall summarize our results here. As in the case of [9, 10] we consider type IIB string theory on $\mathcal{M} \times S^1 \times \widetilde{S}^1$ where $\mathcal{M}$ is either K3 or T^4, and mod out this theory by a $\mathbb{Z}_N$ symmetry group generated by a transformation g that involves $1/N$ unit of shift along the circle S^1 together with an

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order N transformation $\widetilde{g}$ in $\mathcal{M}$. $\widetilde{g}$ is chosen in such a way that the final theory has $\mathcal{N}=4$ supersymmetry. We consider in this theory a configuration with a single D5-brane wrapped on $\mathcal{M}\times S^1$, Q_1 D1-branes wrapped on S^1, a single Kaluza-Klein monopole associated with the circle $\widetilde{S}^1$, $-n/N$ units of momentum along S^1 and J units of momentum along $\widetilde{S}^1$ [3].[1] By making an S-duality transformation, followed by a T-duality along the circle $\widetilde{S}^1$ and a six dimensional string-string duality, we can map this system to an asymmetric $\mathbb{Z}_N$ orbifold of heterotic (for $\mathcal{M}=K3$) [15−17] or type IIA (for $\mathcal{M}=T^4$) [18] string theory on $T^4\times S^1\times\widehat{S}^1$, with $-n/N$ units of momentum along S^1, a single Kaluza-Klein monopole associated with $\widehat{S}^1$, $(Q_1-\beta)$ units of NS 5-brane charge along $T^4\times S^1$, J units of NS 5-brane charge along $T^4\times\widehat{S}^1$ and a single fundamental string wound along S^1 [9]. Here β is the Euler character of $\mathcal{M}$ divided by 24. If Q_e and Q_m denote the electric and magnetic charge vectors in this asymmetric orbifold description,[2] and if $\cdot$ denotes the T-duality invariant inner product in this description, then we have

$$Q_e^2 \equiv Q_e\cdot Q_e = 2n/N, \qquad Q_m^2\equiv Q_m\cdot Q_m = 2(Q_1-\beta), \qquad Q_e\cdot Q_m = J\,. \tag{1.1}$$

We denote by $d(Q_e,Q_m)$ the number of bosonic minus fermionic quarter BPS supermultiplets carrying a given set of charges (Q_e,Q_m), a supermultiplet being considered bosonic (fermionic) if it is obtained by tensoring the basic 64 dimensional quarter BPS supermultiplet with a supersymmetry singlet bosonic (fermionic) state. Our result for $d(Q_e,Q_m)$ is

$$d(Q_e,Q_m)=\frac{1}{N}\int_{\mathcal{C}} d\widetilde{\rho}\, d\widetilde{\sigma}\, d\widetilde{v}\, e^{-\pi i(N\widetilde{\rho}Q_e^2+\widetilde{\sigma}Q_m^2/N+2\widetilde{v}Q_e\cdot Q_m)}\,\frac{1}{\widetilde{\Phi}(\widetilde{\rho},\widetilde{\sigma},\widetilde{v})}\,, \tag{1.2}$$

where $\widetilde{\Phi}$ is a function to be defined below and $\mathcal{C}$ is a three real dimensional subspace of the three complex dimensional space labelled by $(\widetilde{\rho},\widetilde{\sigma},\widetilde{v})$, given by

$$\begin{aligned} &Im\,\widetilde{\rho}=M_1, \quad Im\,\widetilde{\sigma}=M_2, \quad Im\,\widetilde{v}=M_3,\\ &0\le Re\,\widetilde{\rho}\le 1, \quad 0\le Re\,\widetilde{\sigma}\le N, \quad 0\le Re\,\widetilde{v}\le 1\,, \end{aligned} \tag{1.3}$$

where M_1, M_2 and M_3 are large but fixed positive numbers. Alternatively, we can express $d(Q_e,Q_m)$ as

$$d(Q_e,Q_m)=g\left(\frac{N}{2}Q_e^2,\frac{1}{2N}\,Q_m^2,Q_e\cdot Q_m\right)\,, \tag{1.4}$$

where $g(m,n,p)$ are the coefficients of Fourier expansion of the function $1/\widetilde{\Phi}(\widetilde{\rho},\widetilde{\sigma},\widetilde{v})$:

$$\frac{1}{\widetilde{\Phi}(\widetilde{\rho},\widetilde{\sigma},\widetilde{v})}=\sum_{m,n,p} g(m,n,p)\,e^{2\pi i(m\,\widetilde{\rho}+n\,\widetilde{\sigma}+p\,\widetilde{v})}\,. \tag{1.5}$$

[1]Here all the units refer to those in the orbifold theory. Thus for example if S^1 has radius R then in the orbifold theory there will be periodicity under a translation by $2\pi R/N$ along S^1 together with an appropriate transformation on the rest of the conformal field theory. Hence the unit of momentum along S^1 is taken to be N/R.

[2]Unless mentioned otherwise, whenever we refer to electric or magnetic charges or T- or S-duality symmetry of the theory, we shall imply electric or magnetic charges or T- or S-duality symmetry in the asymmetric orbifold description.

In order to define $\widetilde{\Phi}$ we shall have to consider a 2-dimensional (4,4) superconformal σ-model with target space $\mathcal{M}$, modded out by the $\widetilde{\mathbb{Z}}_N$ group generated by the transformation $\widetilde{g}$ described earlier. In this theory we define [19]

$$F^{(r,s)}(\tau,z) \equiv \frac{1}{N} Tr_{RR;\widetilde{g}^r}\left(\widetilde{g}^s(-1)^{F_L+F_R} e^{2\pi i\tau L_0} e^{-2\pi i\bar{\tau}\bar{L}_0} e^{2\pi i F_L z}\right), \qquad 0\le r,s\le N-1\,, \tag{1.6}$$

where Tr denotes trace over all the Ramond-Ramond (RR) sector states twisted by $\widetilde{g}^r$ in the superconformal field theory (SCFT) described above before we project on to $\widetilde{g}$ invariant states, F_L and F_R denote the world-sheet fermion numbers associated with left and right chiral fermions in this SCFT and L_n, $\bar{L}_n$ are the Virasoro generators in this SCFT with additive factors of $-c_L/24$ and $-c_R/24$ included in the definitions of L_0 and $\bar{L}_0$. In this convention the RR sector ground state has $L_0=\bar{L}_0=0$. $F^{(r,s)}(\tau,z)$ can be shown to have an expansion of the form

$$F^{(r,s)}(\tau,z) = \sum_{b=0}^{1}\sum_{\substack{j\in 2\mathbb{Z}+b,\, n\in\mathbb{Z}/N\\ 4n-j^2\ge -b^2}} c_b^{(r,s)}(4n-j^2)\, e^{2\pi i n\tau + 2\pi i j z}\,. \tag{1.7}$$

This defines the coefficients $c_b^{(r,s)}(u)$. We also define

$$Q_{r,s} = N\,\left(c_0^{(r,s)}(0) + 2\,c_1^{(r,s)}(-1)\right)\,, \tag{1.8}$$

and

$$\widetilde{\alpha} = \frac{1}{24N}\,Q_{0,0} - \frac{1}{2N}\sum_{s=1}^{N-1} Q_{0,s}\,\frac{e^{-2\pi i s/N}}{(1-e^{-2\pi i s/N})^2}\,, \qquad \widetilde{\gamma} = \frac{1}{24N}\,Q_{0,0}\,. \tag{1.9}$$

In terms of these coefficients the function $\widetilde{\Phi}$ appearing in (1.2) is given by

$$\begin{aligned}\widetilde{\Phi}(\widetilde{\rho},\widetilde{\sigma},\widetilde{v}) &= e^{2\pi i(\widetilde{\alpha}\widetilde{\rho}+\widetilde{\gamma}\widetilde{\sigma}+\widetilde{v})}\\ &\times\prod_{b=0}^{1}\prod_{r=0}^{N-1}\prod_{\substack{k'\in\mathbb{Z}+\frac{r}{N},\, l\in\mathbb{Z},\, j\in 2\mathbb{Z}+b\\ k',l\ge 0,\, j<0 \text{ for } k'=l=0}} \left(1-\exp\left(2\pi i(k'\widetilde{\sigma}+l\widetilde{\rho}+j\widetilde{v})\right)\right)^{\sum_{s=0}^{N-1} e^{-2\pi i s l/N} c_b^{(r,s)}(4k'l-j^2)}\,.\end{aligned} \tag{1.10}$$

This expression for $\widetilde{\Phi}$, including the values of $\widetilde{\alpha}$ and $\widetilde{\gamma}$, reduces to the ones studied earlier for prime values of N [6−10] except for an overall normalization factor. We have used a new normalization convention for $\widetilde{\Phi}$ to simplify some of the formulæ.

One point about the degeneracy formula given above is worth mentioning. Eqs. (1.2) and (1.4) are equivalent only if the sum over m, n, p in (1.5) are convergent for large imaginary $\widetilde{\rho}$, $\widetilde{\sigma}$ and $\widetilde{v}$. This in particular requires that for fixed m and n the sum over p is bounded from below. By examining the formula (1.10) for $\widetilde{\Phi}$ and the fact that the coefficients $c_b^{(r,s)}(u)$ are non-zero only for $4u\ge -b^2$, we can verify that with the exception of the contribution from the $k'=l=0$ term in this product, the other terms, when expanded in a power series expansion in $e^{2\pi i\widetilde{\rho}}$ and $e^{2\pi i\widetilde{\sigma}}$, does have the form of (1.5) with p bounded from below for fixed m, n. However for the $k'=l=0$ term, which arises from the dynamics of the D1-D5 centre of mass motion in the Kaluza-Klein monopole background and gives a contribution $e^{-2\pi i\widetilde{v}}/(1-e^{-2\pi i\widetilde{v}})^2$ [9], there is an ambiguity in carrying out the series expansion. We could either use the form given above and expand the denominator in a series expansion in $e^{-2\pi i\widetilde{v}}$, or express it in the form $e^{2\pi i\widetilde{v}}/(1-e^{2\pi i\widetilde{v}})^2$ and expand it in a series expansion in $e^{2\pi i\widetilde{v}}$. It was shown in [9] that depending on the angle between S^1 and $\widetilde{S}^1$, only one of these expansions produce the degeneracy formula correctly. The physical spectrum actually changes as this angle passes through $90°$ since at this point the system is only marginally stable. On the other hand our degeneracy formula (1.2) implicitly requires that we expand this factor in powers of $e^{2\pi i\widetilde{v}}$ since only in this case the sum over p in (1.5) is bounded from below for fixed m, n. Thus as it stands the formula is valid for a specific range of values of the angle between S^1 and $\widetilde{S}^1$, which, in the dual asymmetric orbifold description of the system, corresponds to the sign of the axion field. For the other sign of the axion we need to take M_3 to be large and negative to get a correct formula for the degeneracy.

Another point about (1.2) is that it has been derived for special charge vectors Q_e, Q_m in a specific region of the moduli space, – the weakly coupling region in the original description as type IIB string theory on $\mathcal{M}\times S^1\times\widetilde{S}^1/\mathbb{Z}_N$. Thus although we have expressed the formula for $d(Q_e,Q_m)$ in a form that is independent of the asymptotic values of the various moduli fields and as a function of the T-duality invariant combinations Q_e^2, Q_m^2 and $Q_e\cdot Q_m$, it need not have this form in all regions of the moduli space for all charge vectors. In particular the spectrum could change discontinuously across curves of marginal stability as we vary the moduli [20]. Since the duality invariance of the theory only guarantees that the spectrum remains invariant under a simultaneous duality transformation of the moduli and the charge vectors, we cannot invoke duality invariance to find $d(Q_e,Q_m)$ for general charge vectors unless we know the moduli dependence of the formula from other sources.

S-duality invariance of the theory in the asymmetric orbifold description corresponds to global diffeomorphism symmetry associated with the torus $S^1\times\widetilde{S}^1$ in the original description of the theory. This leaves invariant the weak coupling region of the theory, — the region in which the the degeneracy formula (1.2) has been derived. Thus in this region the S-duality transformation should be a symmetry of $d(Q_e,Q_m)$. The problem of verifying this directly however is that we have derived eq. (1.2) for a specific choice of the charge vectors Q_e, Q_m. If we assume that (1.2) is valid for all charge vectors, — at least in the weak coupling region, – then one can verify that this formula is indeed invariant under S-duality transformation. Instead of taking this as a test of S-duality transformation, — which is expected to be true anyway, – we can regard this as an indication that our formula (1.2) for $d(Q_e,Q_m)$ is valid for general charge vectors in the weak coupling region of the original theory.

By performing the integral over $\widetilde{v}$ in (1.2) by picking up residues at the poles of the integrand, and subsequent integral over $\widetilde{\rho}$, $\widetilde{\sigma}$ by a saddle point approximation, we can extract the behaviour of $d(Q_e,Q_m)$ for large charges. The result is that up to first non-leading order, the entropy is given by extremizing a statistical entropy function:

$$-\widetilde{\Gamma}_B(\vec{\tau})=\frac{\pi}{2\tau_2}\,|Q_e+\tau Q_m|^2-\ln g(\tau)-\ln g(-\bar{\tau})-(k+2)\ln(2\tau_2)+\text{constant}+\mathcal{O}(Q^{-2}) \quad (1.11)$$

with respect to real and imaginary parts of the complex variable τ. Here

$$k = \frac{1}{2} \sum_{s=0}^{N-1} c_0^{(0,s)}(0) \tag{1.12}$$

and

$$g(\rho) = e^{2\pi i \widehat{\alpha}\rho} \prod_{n=1}^{\infty} \prod_{r=0}^{N-1} \left(1 - e^{2\pi i r/N} e^{2\pi i n\rho}\right)^{s_r} , \tag{1.13}$$

where

$$s_r = \frac{1}{N} \sum_{s'=0}^{N-1} e^{-2\pi i r s'/N} Q_{0,s'} , \qquad \widehat{\alpha} = \frac{1}{24} Q_{0,0} . \tag{1.14}$$

$|Q_e + \tau Q_m|^2$ appearing in (1.11) is to be interpreted as

$$Q_e^2 + 2\tau_1 Q_e \cdot Q_m + |\tau|^2 Q_m^2 . \tag{1.15}$$

We can calculate the black hole entropy in this theory using the entropy function formalism [21, 22]. The low energy effective action is that of $\mathcal{N} = 4$ supergravity coupled to a certain number of matter multiplets. However stringy corrections give rise to higher derivative terms in the action which include a Gauss-Bonnet term of the form

$$\Delta\mathcal{L} = \phi(\tau, \bar{\tau}) \left\{ R_{G\mu\nu\rho\sigma} R_G^{\mu\nu\rho\sigma} - 4 R_{G\mu\nu} R_G^{\mu\nu} + R_G^2 \right\} , \tag{1.16}$$

where τ denotes the complex structure modulus of the torus $S^1 \times \widetilde{S}^1$. The function $\phi(\tau, \bar{\tau})$ can be calculated using the method of [23] and is given by

$$\phi(\tau, \bar{\tau}) = -\frac{1}{64\pi^2} \left((k+2) \ln \tau_2 + \ln g(\tau) + \ln g(\bar{\tau}) \right) + \text{constant} . \tag{1.17}$$

The entropy of a dyonic black hole, after taking into account corrections due to the Gauss-Bonnet term, is given by the extremum of the black hole entropy function [22]

$$\mathcal{E} = \frac{\pi}{2\tau_2} |Q_e + \tau Q_m|^2 - \ln g(\tau) - \ln g(-\bar{\tau}) - (k+2) \ln(2\tau_2) + \text{constant} + \mathcal{O}(Q^{-2}) . \tag{1.18}$$

Comparing (1.11) and (1.18) we see that the black hole entropy and the statistical entropy agree to this order.[3]

The rest of the paper is organised as follows. Sections 2 and 3 contain general mathematical results which will be useful for studies in the later sections. In section 2 we study in detail some properties of the two dimensional (4,4) superconformal field theory with

[3]The full action contains other four derivative terms besides the Gauss-Bonnet term and hence there is no *a priori* justification for keeping only the Gauss-Bonnet term in the effective action. However at least for $Q_e^2 >> Q_m^2, Q_e \cdot Q_m$ when the coupling constant at the horizon in the asymmetric orbifold description is small, one can show that the Gauss-Bonnet term captures the effect of complete set of four derivative terms [24 – 27]. This is also true if we add to the action the set of all terms related to the curvature squared term via supersymmetry transformation [28, 29]. Thus there is some non-renormalization theorem at work at least for $Q_e^2 >> Q_m^2, Q_e \cdot Q_m$. Our hope is that similar non-renormalization theorems would also hold when all the charges are of the same order.

target space $\mathcal{M}$ modded out by the group $\widetilde{\mathbb{Z}}_N$ generated by $\widetilde{g}$, and various properties of the functions $F^{(r,s)}(\tau, z)$ and their Fourier coefficients $c_b^{(r,s)}(u)$. Section 3 is devoted to studying various properties of the function $\widetilde{\Phi}(\widetilde{\rho}, \widetilde{\sigma}, \widetilde{v})$ and some other related functions which are necessary for studying the duality transformation properties as well as the asymptotic expansion of the statistical entropy. Section 4, which is the main section of this paper, describes the computation of the degeneracy of dyons carrying a given set of charges. As in the analysis of [9, 10] the contribution to the dyon partition function comes from three separate sources, — the dynamics of the Kaluza-Klein monopole, the overall motion of the D1-D5 system in the Kaluza-Klein monopole background and the motion of the D1-brane inside the D5-brane. In section 5 we prove the 'S-duality' invariance of the degeneracy formula. Section 6 describes the asymptotic expansion of the statistical entropy of the system, defined as $\ln d(Q_e, Q_m)$, in the limit of large charges up to first non-leading order. In section 7 we calculate the entropy of a black hole carrying the same charges by taking into account the Gauss-Bonnet term in the low energy effective action and show that the result agrees with the statistical entropy to this order.

2. A class of $(4,4)$ superconformal field theories

In this section we shall introduce a class of (4,4) superconformal field theories which will be useful for later analysis.

Let $\mathcal{M}$ be either a $K3$ or a T^4 manifold, and let $\widetilde{g}$ be an order N discrete symmetry transformation acting on $\mathcal{M}$. We shall choose $\widetilde{g}$ in such a way that it satisfies the following properties (not all of which are independent):

1. We require that in an appropriate complex coordinate system of $\mathcal{M}$, $\widetilde{g}$ preserves the (0,2) and (2,0) harmonic forms of $\mathcal{M}$.

2. Let $\widetilde{\mathbb{Z}}_N$ denote the group generated by $\widetilde{g}$. We shall require that the orbifold $\widehat{\mathcal{M}} = \mathcal{M}/\widetilde{\mathbb{Z}}_N$ has SU(2) holonomy.

3. Let ω_i denote the harmonic 2-forms of $\mathcal{M}$ and
$$I_{ij} = \int_{\mathcal{M}} \omega_i \wedge \omega_j \tag{2.1}$$
denote the intersection matrix of these 2-forms in $\mathcal{M}$. When we diagonalize I we get 3 eigenvalues -1 and a certain number (say P) of the eigenvalues $+1$ ($P = 19$ for $K3$ and 3 for T^4). We call the 2-forms carrying eigenvalue -1 right-handed 2-forms and the 2-forms carrying eigenvalues $+1$ left-handed 2-forms. We shall choose $\widetilde{g}$ such that it leaves invariant all the right-handed 2-forms.

4. The $(4,4)$ superconformal field theory with target space $\mathcal{M}$ has $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ R-symmetry group. We shall require that the transformation $\widetilde{g}$ commutes with the (4,4) superconformal symmetry and the $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ R-symmetry group of the theory. (For $\mathcal{M} = T^4$ the supersymmetry and the R-symmetry groups are bigger, but $\widetilde{g}$ must be such that only the (4,4) superconformal symmetry and the $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ part of the R-symmetry group commute with $\widetilde{g}$.)

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Let us now take an orbifold of this (4,4) superconformal field theory by the group $\widetilde{\mathbb{Z}}_N$ generated by the transformation $\widetilde{g}$, and define [19]

$$F^{(r,s)}(\tau, z) \equiv \frac{1}{N} Tr_{RR;\widetilde{g}^r} \left(\widetilde{g}^s (-1)^{F_L+F_R} e^{2\pi i \tau L_0} e^{-2\pi i \bar{\tau} \bar{L}_0} e^{2\pi i F_L z} \right), \qquad 0 \le r, s \le N-1, \tag{2.2}$$

where Tr denotes trace over all the Ramond-Ramond (RR) sector states twisted by $\widetilde{g}^r$ in the SCFT described above before we project on to $\widetilde{g}$ invariant states, L_n, $\bar{L}_n$ denote the left- and right-moving Virasoro generators and F_L and F_R denote the world-sheet fermion numbers associated with left and right-moving sectors in this SCFT. Equivalently we can identify F_L (F_R) as twice the generator of the $\mathrm{U}(1)_L$ ($\mathrm{U}(1)_R$) subgroup of the $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ R-symmetry group of this conformal field theory.[4] As in [7] we include in the definition of L_0, $\bar{L}_0$ additive factors of $-c_L/24$ and $-c_R/24$ respectively, so that RR sector ground state has $L_0 = \bar{L}_0 = 0$. Due to the insertion of $(-1)^{F_R}$ factor in the trace the contribution to $F^{(r,s)}$ comes only from the $\bar{L}_0 = 0$ states. As a result $F^{(r,s)}$ does not depend on $\bar{\tau}$.

For $\widetilde{g}$ satisfying the conditions described earlier the functions $F^{(r,s)}(\tau, z)$ have the form

$$F^{(r,s)}(\tau, z) = h_0^{(r,s)}(\tau)\, \vartheta_3(2\tau, 2z) + h_1^{(r,s)}(\tau)\, \vartheta_2(2\tau, 2z)\,. \tag{2.3}$$

This follows from the fact that $\vartheta_3(2\tau, 2z)$ and $\vartheta_2(2\tau, 2z)$ are the characters of the $\mathrm{SU}(2)_L$ level 1 current algebra which is a symmetry of this SCFT. The functions $h_b^{(r,s)}(\tau)$ in turn have expansions of the form

$$h_b^{(r,s)}(\tau) = \sum_{n \in \frac{1}{N}\mathbb{Z} - \frac{b^2}{4}} c_b^{(r,s)}(4n) e^{2\pi i n \tau}\,. \tag{2.4}$$

This defines the coefficients $c_b^{(r,s)}(u)$. We shall justify the restriction on the allowed values of n shortly. Using the known expansion of ϑ_3 and ϑ_2:

$$\vartheta_3(2\tau, 2z) = \sum_{j \in 2\mathbb{Z}} e^{2\pi i j z} e^{\pi i \tau j^2/2}, \qquad \vartheta_2(2\tau, 2z) = \sum_{j \in 2\mathbb{Z}+1} e^{2\pi i j z} e^{\pi i \tau j^2/2}, \tag{2.5}$$

we get

$$F^{(r,s)}(\tau, z) = \sum_{b=0}^{1} \sum_{j \in 2\mathbb{Z}+b, n \in \mathbb{Z}/N} c_b^{(r,s)}(4n - j^2) e^{2\pi i n \tau + 2\pi i j z}\,. \tag{2.6}$$

Since in the RR sector the L_0 eigenvalue is ≥ 0 for any state, it follows from (2.3), (2.5) that

$$c_0^{(r,s)}(u) = 0 \quad \text{for } u < 0, \qquad c_1^{(r,s)}(u) = 0 \quad \text{for } u < -1\,. \tag{2.7}$$

$F^{(r,s)}(\tau, z)$ defined in (2.2) may be regarded as the partition function on a torus with modular parameter τ with $\widetilde{g}^s\, e^{2\pi i F_L z}$ twist along the b-cycle and $\widetilde{g}^r$ twist along the a-cycle. If (σ_1, σ_2) denote the coordinates of this torus, each with period 1, then under

[4]At this stage we are describing an abstract conformal field theory without connecting it to string theory. In all cases where we use this conformal field theory to describe a fundamental string world-sheet theory or world-volume theory of some soliton, we shall use the Green-Schwarz formulation. Thus the world-sheet fermion number of this SCFT will represent the space-time fermion number in string theory.

$\sigma_1 \to -\sigma_1$, $\sigma_2 \to -\sigma_2$, the quantum numbers r and s change sign and also $z \to -z$. Thus $F^{(r,s)}(\tau, z) = F^{(-r,-s)}(\tau, -z)$. It then follows from (2.4)), ((2.6), that

$$h_b^{(r,s)}(\tau) = h_b^{(-r,-s)}(\tau)\,, \qquad c_b^{(r,s)}(u) = c_b^{(-r,-s)}(u)\,. \tag{2.8}$$

Furthermore, since under $(\sigma_1, \sigma_2) \to (\sigma_1 + \sigma_2, \sigma_2)$ the modular parameter τ gets shifted by 1 and $(r, s) \to (r, s + r)$, we must have $F^{(r,s+r)}(\tau + 1, z) = F^{(r,s)}(\tau, z)$. Since (r, s) are defined modulo N we get $F^{(r,s)}(\tau, z) = F^{(r,s)}(\tau + N, z)$. This is the physical origin of the restriction $n \in \mathbb{Z}/N$ in (2.6) and $n \in \mathbb{Z}/N - b^2/4$ in (2.4).

The $n = 0$ terms in the expansion (2.6) is given by the contribution to (2.2) from the RR sector states with $L_0 = \bar{L}_0 = 0$. For $r = 0$, i.e. in the untwisted sector, these states are in one to one correspondence with harmonic (p, q) forms on $\mathcal{M}$, with $(p - 1)$ and $(q - 1)$ measuring the quantum numbers F_L and F_R [30, 31]. Thus $N\, c_0^{(0,s)}(0)$, being $N\times$ the coefficient of the $n = 0$, $j = 0$ term in (2.6), measures the number of harmonic $(1, q)$ forms weighted by $(-1)^{q-1}\widetilde{g}^s$, and $N\, c_1^{(0,s)}(-1)$, being $N\times$ the coefficient of the $n = 0$, $j = -1$ (or $j = 1$) term in (2.6), measures the number of harmonic $(0, q)$ (or $(2, q)$) forms weighted by $(-1)^q\widetilde{g}^s$. If $\mathcal{M} = K3$ then the only $(0, q)$ forms are $(0, 0)$ and $(0, 2)$ forms both of which are invariant under $\widetilde{g}$. Thus we have

$$c_1^{(0,s)}(-1) = \frac{2}{N} \quad \text{for } \mathcal{M} = K3\,. \tag{2.9}$$

On the other hand for $\mathcal{M} = T^4$ one can represent the explicit action of $\widetilde{g}$ in an appropriate complex coordinate system (z^1, z^2) as

$$dz^1 \to e^{2\pi i/N} dz^1, \quad dz^2 \to e^{-2\pi i/N} dz^2\,, \quad d\bar{z}^1 \to e^{-2\pi i/N} d\bar{z}^1, \quad d\bar{z}^2 \to e^{2\pi i/N} d\bar{z}^2\,, \tag{2.10}$$

so that it preserves the $(2, 0)$ and $(0, 2)$ forms $dz^1 \wedge dz^2$ and $d\bar{z}^1 \wedge d\bar{z}^2$. Using this one can work out its action on all the 2-, 3- and 4-forms:

$$\begin{aligned}
&dz^1 \wedge dz^2 \to dz^1 \wedge dz^2, \quad dz^1 \wedge d\bar{z}^1 \to dz^1 \wedge d\bar{z}^1, \quad dz^1 \wedge d\bar{z}^2 \to e^{4\pi i/N}\, dz^1 \wedge d\bar{z}^2,\\
&d\bar{z}^1 \wedge d\bar{z}^2 \to d\bar{z}^1 \wedge d\bar{z}^2, \quad dz^2 \wedge d\bar{z}^2 \to dz^2 \wedge d\bar{z}^2, \quad d\bar{z}^1 \wedge dz^2 \to e^{-4\pi i/N}\, d\bar{z}^1 \wedge dz^2\,,
\end{aligned} \tag{2.11}$$

$$\begin{aligned}
&dz^1 \wedge dz^2 \wedge d\bar{z}^1 \to e^{-2\pi i/N}\, dz^1 \wedge dz^2 \wedge d\bar{z}^1, \quad dz^1 \wedge dz^2 \wedge d\bar{z}^2 \to e^{2\pi i/N}\, dz^1 \wedge dz^2 \wedge d\bar{z}^2,\\
&d\bar{z}^1 \wedge d\bar{z}^2 \wedge dz^1 \to e^{2\pi i/N}\, d\bar{z}^1 \wedge d\bar{z}^2 \wedge dz^1, \quad d\bar{z}^1 \wedge d\bar{z}^2 \wedge dz^2 \to e^{-2\pi i/N}\, d\bar{z}^1 \wedge d\bar{z}^2 \wedge dz^2,
\end{aligned} \tag{2.12}$$

$$dz^1 \wedge dz^2 \wedge d\bar{z}^1 \wedge d\bar{z}^2 \to dz^1 \wedge dz^2 \wedge d\bar{z}^1 \wedge d\bar{z}^2\,. \tag{2.13}$$

This shows that the $(0, 0)$ and $(0, 2)$ forms are invariant under $\widetilde{g}$ but the two $(0, 1)$ forms carry $\widetilde{g}$ eigenvalues $\pm 2\pi/N$. Thus we have

$$c_1^{(0,s)}(-1) = \frac{1}{N}\left(2 - e^{2\pi i s/N} - e^{-2\pi i s/N}\right) \quad \text{for } \mathcal{M} = T^4\,. \tag{2.14}$$

(2.11) also shows that $\widetilde{g}$ acts trivially on four of the 2-forms, and acts as a rotation by $4\pi/N$ in the two dimensional subspace spanned by the other two 2-forms. By writing the 2-forms

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in the real basis one can easily verify that the 2-forms which transform non-trivially under $\widetilde{g}$ correspond to left-handed 2-forms. These results will be useful later.

Another useful set of results emerges by taking the $z \to 0$ limit of eqs. (2.2) and (2.6). This gives

$$\sum_{b=0}^{1} \sum_{j\in 2\mathbb{Z}+b, n\in\mathbb{Z}/N} c_b^{(r,s)}(4n - j^2) e^{2\pi i n\tau} = \frac{1}{N} Q_{r,s} , \tag{2.15}$$

where

$$Q_{r,s} = Tr_{RR;\widetilde{g}^r} \left(\widetilde{g}^s (-1)^{F_L+F_R} e^{2\pi i\tau L_0} e^{-2\pi i\bar{\tau}\bar{L}_0} \right), \qquad 0 \le r, s \le N-1 . \tag{2.16}$$

$Q_{r,s}$ is independent of τ and $\bar{\tau}$ since the $(-1)^{F_L+F_R}$ insertion in the trace makes the contribution from the $(L_0, \bar{L}_0) \neq (0,0)$ states cancel. Thus (2.15) gives

$$\sum_{b=0}^{1} \sum_{j\in 2\mathbb{Z}+b} c_b^{(r,s)}(4n - j^2) = \frac{1}{N} Q_{r,s}\, \delta_{n,0} . \tag{2.17}$$

Setting $n = 0$ in the above equation and using eq. (2.7) we get

$$Q_{r,s} = N \left(c_0^{(r,s)}(0) + 2\, c_1^{(r,s)}(-1) \right) . \tag{2.18}$$

For $r = 0$, i.e. in the untwisted sector, the trace in (2.16) reduces to a sum over the harmonic forms of $\mathcal{M}$. Since $F_L + F_R$ is mapped to the degree of the harmonic form, $Q_{0,s}$ has the interpretation of trace of $(-1)^p \widetilde{g}^s$ over the harmonic p-forms of $\mathcal{M}$. In particular we have

$$Q_{0,0} = \chi(\mathcal{M}) , \tag{2.19}$$

where $\chi(\mathcal{M})$ denotes the Euler number of $\mathcal{M}$.

For later use we shall define

$$\widehat{F}^{(r,s)}(\tau, z) = \frac{1}{N} \sum_{s'=0}^{N-1} \sum_{r'=0}^{N-1} e^{-2\pi i r s'/N} e^{2\pi i r' s/N}\, F^{(r',s')}(\tau, z) . \tag{2.20}$$

$\widehat{F}^{(r,s)}(\tau, z)$ satisfies properties similar to that of $F^{(r,s)}(\tau, z)$. In particular we have the relations:

$$\widehat{F}^{(r,s)}(\tau, z) = \sum_{b=0}^{1} \sum_{j\in 2\mathbb{Z}+b, n\in\mathbb{Z}/N} \widehat{c}_b^{(r,s)}(4n - j^2) e^{2\pi i n\tau + 2\pi i j z} , \tag{2.21}$$

where

$$\widehat{c}_b^{(r,s)}(u) = \frac{1}{N} \sum_{r'=0}^{N-1} \sum_{s'=0}^{N-1} e^{2\pi i (s r' - r s')/N} c_b^{(r',s')}(u) . \tag{2.22}$$

We also have the analog of eq. (2.17)

$$\sum_{b=0}^{1} \sum_{j\in 2\mathbb{Z}+b} \widehat{c}_b^{(r,s)}(4n - j^2) e^{2\pi i n\tau} = \frac{1}{N} \widehat{Q}_{r,s}\, \delta_{n,0} , \tag{2.23}$$

where

$$\widehat{Q}_{r,s} = \frac{1}{N} \sum_{s'=0}^{N-1} \sum_{r'=0}^{N-1} e^{2\pi i (s r' - r s')/N}\, Q_{r',s'} . \tag{2.24}$$

3. Siegel modular forms from threshold integrals

In this section we shall prove various properties of $\widetilde{\Phi}$ defined in (1.10) by relating it to a 'threshold integral' [32]. We begin by defining:

$$\Omega = \begin{pmatrix} \rho & v \\ v & \sigma \end{pmatrix}, \tag{3.1}$$

and

$$\begin{aligned} \frac{1}{2}p_R^2 &= \frac{1}{4\det Im\Omega}|-m_1\rho + m_2 + n_1\sigma + n_2(\sigma\rho - v^2) + jv|^2, \\ \frac{1}{2}p_L^2 &= \frac{1}{2}p_R^2 + m_1n_1 + m_2n_2 + \frac{1}{4}j^2, \end{aligned} \tag{3.2}$$

where ρ, σ and v are three complex variables. We now consider the 'threshold integrals'

$$\widetilde{\mathcal{I}}(\rho,\sigma,v) = \sum_{r,s=0}^{N-1}\sum_{b=0}^{1}\widetilde{\mathcal{I}}_{r,s,b}\,, \qquad \widehat{\mathcal{I}}(\rho,\sigma,v) = \sum_{r,s=0}^{N-1}\sum_{b=0}^{1}\widehat{\mathcal{I}}_{r,s,b}\,, \tag{3.3}$$

where

$$\widetilde{\mathcal{I}}_{r,s,b} = \int_{\mathcal{F}}\frac{d^2\tau}{\tau_2}\sum_{\substack{m_1,m_2,n_2\in\mathbb{Z}\\ n_1\in\mathbb{Z}+\frac{r}{N},j\in 2\mathbb{Z}+b}} q^{p_L^2/2}\bar{q}^{p_R^2/2}e^{2\pi i m_1 s/N}h_b^{(r,s)}(\tau)\,, \tag{3.4}$$

and

$$\widehat{\mathcal{I}}_{r,s,b} = \int_{\mathcal{F}}\frac{d^2\tau}{\tau_2}\sum_{\substack{m_1,n_1\in\mathbb{Z},m_2\in\mathbb{Z}/N\\ n_2\in N\mathbb{Z}+r,j\in 2\mathbb{Z}+b}} q^{p_L^2/2}\bar{q}^{p_R^2/2}e^{2\pi i m_2 s}h_b^{(r,s)}(\tau)\,, \tag{3.5}$$

with

$$q \equiv e^{2\pi i\tau}\,. \tag{3.6}$$

Let us now introduce another set of variables $(\widetilde{\rho},\widetilde{\sigma},\widetilde{v})$ related to (ρ,σ,v) via the relations

$$\widetilde{\rho} = \frac{1}{N}\frac{1}{2v-\rho-\sigma}, \qquad \widetilde{\sigma} = N\frac{v^2-\rho\sigma}{2v-\rho-\sigma}, \qquad \widetilde{v} = \frac{v-\rho}{2v-\rho-\sigma}\,, \tag{3.7}$$

or equivalently,

$$\rho = \frac{\widetilde{\rho}\widetilde{\sigma}-\widetilde{v}^2}{N\widetilde{\rho}}, \qquad \sigma = \frac{\widetilde{\rho}\widetilde{\sigma}-(\widetilde{v}-1)^2}{N\widetilde{\rho}}, \qquad v = \frac{\widetilde{\rho}\widetilde{\sigma}-\widetilde{v}^2+\widetilde{v}}{N\widetilde{\rho}}\,. \tag{3.8}$$

We also define

$$\widetilde{\Omega} = \begin{pmatrix} \widetilde{\rho} & \widetilde{v} \\ \widetilde{v} & \widetilde{\sigma} \end{pmatrix}. \tag{3.9}$$

By relabelling the indices m_1, m_2, n_1, n_2 in eqs. (3.4)–(3.5) one can easily prove the relations

$$\widehat{\mathcal{I}}(\rho,\sigma,v) = \widetilde{\mathcal{I}}(\widetilde{\rho},\widetilde{\sigma},\widetilde{v})\,. \tag{3.10}$$

In the same way one can show that under a transformation of the form

$$\Omega \to (A\Omega+B)(C\Omega+D)^{-1}\,, \tag{3.11}$$

$\widehat{\mathcal{I}}(\rho,\sigma,v)$ remains invariant for the following choices of the matrices A, B, C, D:

$$
\begin{aligned}
\begin{pmatrix} A & B \\ C & D \end{pmatrix} &= \begin{pmatrix} a & 0 & b & 0 \\ 0 & 1 & 0 & 0 \\ c & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad ad-bc=1, \quad c=0 \bmod N, \quad a,d=1 \bmod N \\
\begin{pmatrix} A & B \\ C & D \end{pmatrix} &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \\
\begin{pmatrix} A & B \\ C & D \end{pmatrix} &= \begin{pmatrix} 1 & 0 & 0 & \mu \\ \lambda & 1 & \mu & 0 \\ 0 & 0 & 1 & -\lambda \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \lambda,\mu\in\mathbb{Z}.
\end{aligned}
\tag{3.12}
$$

The group of transformations generated by these matrices is a subgroup of the Siegel modular group $\mathrm{Sp}(2,\mathbb{Z})$; we shall denote this subgroup by $\widehat{G}$.[5] Via eq. (3.10) this also induces a group of symmetry transformations of $\widetilde{\mathcal{I}}(\widetilde\rho,\widetilde\sigma,\widetilde v)$; we shall denote this group by $\widetilde{G}$.

We can now follow the procedure of [7] to evaluate the integrals $\widetilde{\mathcal{I}}$ and $\widehat{\mathcal{I}}$. Since the procedure is identical to that in [7], we shall only quote the final results:

$$
\widetilde{\mathcal{I}}(\rho,\sigma,v) = -2\ln\left[(\det\,\mathrm{Im}\Omega)^k\right] - 2\ln\widetilde{\Phi}(\rho,\sigma,v) - 2\ln\bar{\widetilde{\Phi}}(\rho,\sigma,v) + \text{constant} \tag{3.13}
$$

and

$$
\widehat{\mathcal{I}}(\rho,\sigma,v) = -2\ln\left[(\det\,\mathrm{Im}\Omega)^k\right] - 2\ln\widehat{\Phi}(\rho,\sigma,v) - 2\ln\bar{\widehat{\Phi}}(\rho,\sigma,v) + \text{constant} \tag{3.14}
$$

where

$$
k = \frac{1}{2}\sum_{s=0}^{N-1} c_0^{(0,s)}(0)\,, \tag{3.15}
$$

$$
\widetilde{\Phi}(\rho,\sigma,v) = e^{2\pi i(\widetilde\alpha\rho+\widetilde\gamma\sigma+v)} \prod_{b=0}^{1}\prod_{r=0}^{N-1} \prod_{\substack{k'\in\mathbb{Z}+\frac{r}{N},\, l\in\mathbb{Z},\, j\in 2\mathbb{Z}+b \\ k',l\ge 0,\, j<0 \text{ for } k'=l=0}} \left(1-e^{2\pi i(k'\sigma+l\rho+jv)}\right)^{\sum_{s=0}^{N-1} e^{-2\pi i s l/N} c_b^{(r,s)}(4k'l-j^2)} \tag{3.16}
$$

and

$$
\widehat{\Phi}(\rho,\sigma,v) = e^{2\pi i(\widehat\alpha\rho+\widehat\gamma\sigma+v)} \prod_{b=0}^{1}\prod_{r,s=0}^{N-1} \prod_{\substack{(k',l)\in\mathbb{Z},\, j\in 2\mathbb{Z}+b \\ k',l\ge 0,\, j<0 \text{ for } k'=l=0}} \left\{1-e^{2\pi i r/N}\, e^{2\pi i(k'\sigma+l\rho+jv)}\right\}^{\widehat{c}_b^{(r,s)}(4k'l-j^2)} \tag{3.17}
$$

[5]For prime values of N the group $\widehat{G}$ is identical to the group G introduced in [6].

with $\widehat{c}_b^{(r,s)}(u)$ given in (2.22), and

$$\widetilde{\alpha} = \frac{1}{24N}\, Q_{0,0} - \frac{1}{2N} \sum_{s=1}^{N-1} Q_{0,s}\, \frac{e^{-2\pi i s/N}}{(1-e^{-2\pi i s/N})^2}\,, \qquad \widetilde{\gamma} = \frac{1}{24N}\, Q_{0,0},$$
$$\widehat{\alpha} = \widehat{\gamma} = \frac{1}{24}\, Q_{0,0}\,. \tag{3.18}$$

The quantities $Q_{r,s}$ have been defined in eqs. (2.16). In arriving at (3.16)–(3.17) we have used the relations (2.8), (2.17) and also (2.9), (2.14). The constant k defined in (3.15) has the interpretation of being half the number of $\widetilde{g}$ invariant $(1,q)$ forms weighted by $(-1)^{q+1}$.

It now follows from (3.10), (3.13) and (3.14) that

$$\widetilde{\Phi}(\widetilde{\rho}, \widetilde{\sigma}, \widetilde{v}) = C_1\, (2v - \rho - \sigma)^k\, \widehat{\Phi}(\rho, \sigma, v) \tag{3.19}$$

where C_1 is a constant. Furthermore given the invariance of $\widetilde{\mathcal{I}}$ and $\widehat{\mathcal{I}}$ under the groups $\widetilde{G}$ and $\widehat{G}$, it follows that $\widetilde{\Phi}$ and $\widehat{\Phi}$ transform as modular forms of weight k under the groups $\widetilde{G}$ and $\widehat{G}$ respectively.

From (3.17), (2.9), (2.14) and (2.23) it is easy to see that for small v

$$\widehat{\Phi}(\rho, \sigma, v) = -4\pi^2\, v^2\, g(\rho)\, g(\sigma) + \mathcal{O}(v^4) \tag{3.20}$$

where

$$g(\rho) = e^{2\pi i \widehat{\alpha} \rho} \prod_{n=1}^{\infty} \prod_{r=0}^{N-1} \left(1 - e^{2\pi i r/N} e^{2\pi i n \rho}\right)^{s_r}\,, \tag{3.21}$$

$$s_r = \frac{1}{N} \sum_{s=0}^{N-1} \widehat{Q}_{r,s} = \frac{1}{N} \sum_{s'=0}^{N-1} e^{-2\pi i r s'/N}\, Q_{0,s'}\,. \tag{3.22}$$

Eq. (3.19) then gives, for small v, i.e. small $\widetilde{\rho}\widetilde{\sigma} - \widetilde{v}^2 + \widetilde{v}$,

$$\widetilde{\Phi}(\widetilde{\rho}, \widetilde{\sigma}, \widetilde{v}) = -4\pi^2\, C_1\, (2v - \rho - \sigma)^k\, v^2 g(\rho)\, g(\sigma) + \mathcal{O}(v^4)\,. \tag{3.23}$$

s_r has the interpretation of being the number of harmonic p-forms in $\mathcal{M}$ with $\widetilde{g}$ eigenvalue $e^{2\pi i r/N}$ weighted by $(-1)^p$. Thus it is an integer.

We can determine the locations of the other zeroes and poles of $\widetilde{\Phi}(\widetilde{\rho}, \widetilde{\sigma}, \widetilde{v})$ by identifying the logarithmic singularities of $\widetilde{\mathcal{I}}(\widetilde{\rho}, \widetilde{\sigma}, \widetilde{v})$ as in [9]. One finds that $\widetilde{\Phi}(\widetilde{\rho}, \widetilde{\sigma}, \widetilde{v})$ has possible zeroes at

$$\left(n_2(\widetilde{\sigma}\widetilde{\rho} - \widetilde{v}^2) + j\widetilde{v} + n_1\widetilde{\sigma} - \widetilde{\rho} m_1 + m_2\right) = 0 \tag{3.24}$$

for $m_1, m_2, n_2 \in \mathbb{Z}$, $n_1 \in \frac{1}{N}\mathbb{Z}$, $j \in 2\mathbb{Z}+1$, $m_1 n_1 + m_2 n_2 + \frac{j^2}{4} = \frac{1}{4}$.

The order of the zero is given by

$$\sum_{s=0}^{N-1} e^{2\pi i m_1 s/N} c_1^{(r,s)}(-1), \qquad r = N\, n_1 \bmod N\,. \tag{3.25}$$

For $N \geq 5$ there are additional possible zeroes of $\widetilde{\Phi}(\widetilde{\rho}, \widetilde{\sigma}, \widetilde{v})$ at

$$\left(n_2(\widetilde{\sigma}\widetilde{\rho} - \widetilde{v}^2) + j\widetilde{v} + n_1\widetilde{\sigma} - \widetilde{\rho} m_1 + m_2\right) = 0 \tag{3.26}$$

for $m_1, m_2, n_2 \in \mathbb{Z}$, $n_1 \in \frac{1}{N}\mathbb{Z}$, $j \in 2\mathbb{Z}+1$, $m_1 n_1 + m_2 n_2 + \frac{j^2}{4} = \frac{1}{4} - \frac{1}{N}$.

The order of the zero is

$$\sum_{s=0}^{N-1} e^{2\pi i m_1 s/N}\, c_1^{(r,s)} \left(-1 + \frac{4}{N}\right), \qquad r = N n_1 \bmod N\,. \tag{3.27}$$

(3.25) has the interpretation as the number of $\widetilde{g}^r$ twisted states with $\widetilde{g}$ eigenvalue $e^{-2\pi i m_1/N}$, $F_L = 1$ (or $F_L = -1$) and $L_0 = \bar{L}_0 = 0$, weighted by $(-1)^{F_L+F_R}$. (3.27) has the interpretation as the number of $\widetilde{g}^r$ twisted states with $\widetilde{g}$ eigenvalue $e^{-2\pi i m_1/N}$, $F_L = 1$ (or $F_L = -1$), $L_0 = 1/N$ and $\bar{L}_0 = 0$, weighted by $(-1)^{F_L+F_R}$. Thus both numbers are integers.

4. Dyon partition function

We now consider type IIB string theory compactified on $\mathcal{M} \times S^1 \times \widetilde{S}^1$, $\mathcal{M}$ being either K3 or T^4. For definiteness we shall label S^1 and $\widetilde{S}^1$ by coordinates with period 2π. We then take an orbifold of this theory by a discrete $\mathbb{Z}_N$ transformation generated by a transformation g, where g involves a $2\pi/N$ translation along S^1 together with an order N transformation $\widetilde{g}$ on $\mathcal{M}$ described in section 2. Due to the properties of $\widetilde{g}$ described earlier, the resulting orbifold preserves all the supersymmetries of type IIB string theory compactified on $K3 \times S^1 \times \widetilde{S}^1$. Thus if $\mathcal{M}$ is T^4 then the orbifolding breaks half of the supersymmetries whereas for $\mathcal{M} = K3$ the orbifolding preserves all the supersymmetries. By making an S-duality transformation of type IIB string theory, followed by a T-duality transformation on the circle $\widetilde{S}^1$ and a string-string duality transformation relating type IIA string theory on $\mathcal{M}$ to type IIA or heterotic string theory on T^4, one can obtain a dual description of these theories as asymmetric orbifolds of heterotic on T^6 for $\mathcal{M} = K3$ and asymmetric orbifolds of type IIB on T^6 for $\mathcal{M} = T^4$. In this description all the space-time supersymmetries arise from the right-moving sector of the fundamental string world-sheet [9, 10]. We shall choose the coordinates along the circles S^1 and $\widetilde{S}^1$ such that before the orbifold projection they have periodicity 2π.

In the original description of the theory as type IIB on $(\mathcal{M} \times S^1 \times \widetilde{S}^1)/\mathbb{Z}_N$ we consider a system containing a single D5-brane wrapped on $\mathcal{M} \times S^1/\mathbb{Z}_N$, Q_1 D1-branes wrapped on $S^1/\mathbb{Z}_N$, momentum $-n$ along S^1, momentum J along $\widetilde{S}^1$ and a Kaluza-Klein monopole associated with the compact circle $\widetilde{S}^1$. In the dual asymmetric orbifold description, the quantum numbers n and the single Kaluza-Klein monopole charge in the original theory appear as momentum $-n$ and single fundamental string wound along S^1. Hence they form part of the electric charge vector Q_e. On the other hand the D1-brane, D5-brane and the momentum along $\widetilde{S}^1$ in the original theory correspond to a single Kaluza-Klein monopole and $(Q_1 - \beta)$ H-monopoles associated with the dual circle of $\widetilde{S}^1$, and J H-monopoles associated with the circle $S^1/\mathbb{Z}_N$ in the dual theory, where

$$\beta = \frac{1}{24}\chi(\mathcal{M})\,, \tag{4.1}$$

$\chi(\mathcal{M})$ being the Euler character of $\mathcal{M}$. Thus they form part of the magnetic charge vector Q_m, and we have [9, 10]

$$Q_e^2 \equiv Q_e \cdot Q_e = 2n/N, \qquad Q_m^2 \equiv Q_m \cdot Q_m = 2(Q_1 - \beta), \qquad Q_e \cdot Q_m = J\,, \tag{4.2}$$

where $\cdot$ denotes T-duality invariant inner product. The $-\beta$ term in the expression for Q_m^2 reflects the fact that a D5-brane wrapped on $\mathcal{M}$ carries $-\chi(\mathcal{M})/24$ unit of D1-brane charge.

The S-duality symmetry of the theory in the asymmetric orbifold description is related to the global diffeomorphism symmetry of the torus $S^1 \times \widetilde{S}^1$ in the original description. More precisely it is the subgroup of this global diffeomorphism group which leaves invariant $2\pi/N$ translation along S^1, and is represented by the $\Gamma_1(N)$ matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ satisfying

$$ad - bc = 1, \quad a, d \in 1 + N\mathbb{Z}, \quad c \in N\mathbb{Z}, \quad b \in \mathbb{Z}\,. \tag{4.3}$$

The duality transformation acts on the electric and the magnetic charge vectors as

$$\begin{pmatrix} Q_e \\ Q_m \end{pmatrix} \to \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} Q_e \\ Q_m \end{pmatrix}. \tag{4.4}$$

Our goal is to find the spectrum of 1/4 BPS states with charge quantum numbers (Q_e, Q_m). Since these states break 12 of the 16 supersymmetry generators of the theory, quantization of the fermionic zero modes associated with the broken supersymmetry generators gives rise to $2^6 = 64$-fold degeneracy, with equal number of bosonic and fermionic states. This 64-fold degeneracy is associated with the size of the 1/4-BPS supermultiplet, and a generic 1/4 BPS state is obtained by tensoring the basic supermultiplet containing 64 states with helicity ranging from $-\frac{3}{2}$ to $\frac{3}{2}$ with a supersymmetry invariant state which could be either bosonic of fermionic. We shall call such supermultiplets bosonic and fermionic supermultiplets respectively, and denote by $d(Q_e, Q_m)$ the number of 1/4 BPS bosonic supermultiplets minus the number of 1/4 BPS fermionic supermultiplets for a given set of charges (Q_e, Q_m).

Another description of $d(Q_e, Q_m)$, equivalent to the one given above, is as follows [23]. If h denotes the helicity of a state, then

$$d(Q_e, Q_m) = \frac{2^6}{6!}\, Tr\left((-1)^{2h} h^6\right), \tag{4.5}$$

where the trace is taken over all 1/4 BPS states with charge quantum numbers (Q_e, Q_m).

In the present example the charges (Q_e, Q_m) are labelled by the set of integers Q_1, n and J together with the number of D5-branes along $\mathcal{M} \times S^1$ and the number of Kaluza-Klein monopoles associated with the circle $\widetilde{S}^1$ in the original description, both of which have been taken to be 1. We shall denote by $h(Q_1, n, J)$ the number of bosonic supermultiplets minus the number of fermionic supermultiplets carrying quantum numbers (Q_1, n, J). Computation of $h(Q_1, n, J)$ is best done in the weak coupling limit of the original description of the system where the quantum numbers n and J arise from three different sources [9]: the excitations of the Kaluza-Klein monopole which can carry certain amount of momentum $-l_0'$ along S^1, the overall motion of the D1-D5 system in the background of the Kaluza-Klein monopole which can carry certain amount of momentum $-l_0$ along S^1 and j_0 along $\widetilde{S}^1$ and the motion of the D1-branes in the plane of the D5-brane carrying total momentum $-L$ along S^1 and J' along $\widetilde{S}^1$. Thus we have

$$l_0' + l_0 + L = n, \qquad j_0 + J' = J\,. \tag{4.6}$$

Let

$$f(\widetilde{\rho},\widetilde{\sigma},\widetilde{v}) = \sum_{Q_1,n,J} h(Q_1,n,J)e^{2\pi i(\widetilde{\rho}n+\widetilde{\sigma}Q_1/N+\widetilde{v}J)}\,, \tag{4.7}$$

denote the partition function of the system. Then in the weak coupling limit we can ignore the interaction between the three different sets of degrees of freedom described above, and $f(\widetilde{\rho},\widetilde{\sigma},\widetilde{v})$ is obtained as a product of three separate partition functions:

$$\begin{aligned} f(\widetilde{\rho},\widetilde{\sigma},\widetilde{v}) = \frac{1}{64}&\sum_{Q_1,L,J'} d_{D1}(Q_1,L,J')e^{2\pi i(\widetilde{\sigma}Q_1/N+\widetilde{\rho}L+\widetilde{v}J')} \\ &\left(\sum_{l_0,j_0} d_{\rm CM}(l_0,j_0)e^{2\pi i l_0\widetilde{\rho}+2\pi i j_0\widetilde{v}}\right)\left(\sum_{l_0'} d_{\rm KK}(l_0')e^{2\pi i l_0'\widetilde{\rho}}\right)\,, \end{aligned} \tag{4.8}$$

where $d_{D1}(Q_1,L,J')$ is the degeneracy of Q_1 D1-branes moving in the plane of the D5-brane carrying momenta $(-L,J')$ along $(S^1,\widetilde{S}^1)$, $d_{\rm CM}(l_0,j_0)$ is the degeneracy associated with the overall motion of the D1-D5 system in the background of the Kaluza-Klein monopole carrying momenta $(-l_0,j_0)$ along $(S^1,\widetilde{S}^1)$ and $d_{\rm KK}(l_0')$ denotes the degeneracy associated with the excitations of a Kaluza-Klein monopole carrying momentum $-l_0'$ along S^1. The factor of 1/64 in (4.8) accounts for the fact that a single 1/4 BPS supermultiplet has 64 states. In each of these sectors we count the degeneracy weighted by $(-1)^F$ with F denoting space-time fermion number of the state, except for the parts obtained by quantizing the fermion zero-modes associated with the broken supersymmetry generators. Since a Kaluza-Klein monopole in type IIB string theory on $K3\times S^1\times\widetilde{S}^1$ breaks 8 of the 16 supersymmetries, quantization of the fermion zero modes associated with the broken supersymmetry generators give rise to a 16-fold degeneracy which appears as a factor in $d_{\rm KK}(l_0')$. Furthermore since a D1-D5 system in the background of a Kaluza-Klein monopole in type IIB on $K3\times S^1\times\widetilde{S}^1$ breaks 4 of the 8 remaining supersymmetry generators, we get a 4-fold degeneracy from the associated fermion zero modes appearing as a factor in $d_{\rm CM}(l_0,j_0)$. This factor of 16×4 cancel the 1/64 factor in (4.8). After separating out this factor, we count the contribution to the degeneracy from the rest of the degrees of freedom weighted by a factor of $(-1)^F$.

We shall now compute each of the three pieces, $d_{\rm KK}(l_0')$, $d_{\rm CM}(l_0,j_0)$ and $d_{D1}(Q_1,L,J')$ separately.

4.1 Counting states of the Kaluza-Klein monopole

We consider type IIB string theory in the background $\mathcal{M}\times TN\times S^1$ where TN denotes Taub-NUT space. This describes type IIB string theory compactified on $\mathcal{M}\times S^1\times\widetilde{S}^1$ in the presence of a Kaluza-Klein monopole, with $\widetilde{S}^1$ identified with the asymptotic circle of the Taub-NUT space. We now take an orbifold of the theory by a $\mathbb{Z}_N$ group generated by the transformation g. Our goal is to compute the degeneracy of the half-BPS states of the Kaluza-Klein monopole carrying momentum $-l_0'$ along S^1.

The world-volume of the Kaluza-Klein monopole is 5+1 dimensional with the five spatial directions lying along $\mathcal{M}\times S^1$. By taking the size of $\mathcal{M}$ to be much smaller than that of S^1 we shall regard this as a 1+1 dimensional theory, obtained by dimensional reduction of the original 5+1 dimensional theory on $\mathcal{M}$. Since the supersymmetry generators of type IIB string theory on K3 are chiral, the world-volume supersymmetry on the Kaluza-Klein monopole will also be chiral, acting on the right-moving degrees of freedom of the 1+1 dimensional field theory. Thus the BPS states of the Kaluza-Klein monopole will correspond to states in this field theory where the right-moving oscillators are in their ground state. In order to count these states we first need to determine the low energy limit of this world-volume theory. Since a Kaluza-Klein monopole has three transverse directions, there are three non-chiral massless bosonic fields on the world-sheet associated with oscillations in these transverse directions. Since Taub-NUT space has a normalizable self-dual harmonic 2-form [33, 34], we can get two additional non-chiral scalar modes on the world-sheet of the Kaluza-Klein monopole by reducing the two 2-form fields of type IIB string theory along this harmonic 2-form. Finally, the self-dual four form field of type IIB theory, reduced along the tensor product of the harmonic 2-form on TN and a harmonic 2-form on $\mathcal{M}$, can give rise to a chiral scalar field on the world-sheet. The chirality of the scalar field is correlated with whether the corresponding harmonic 2-form on $\mathcal{M}$ is self-dual or anti-self-dual. This gives 3 right-moving and P left-moving scalars where $P = 3$ for $\mathcal{M} = T^4$ and 19 for $\mathcal{M} = K3$. Thus we have altogether 8 right-moving scalar fields and $P + 5$ left-moving scalar fields on the world-volume of the Kaluza-Klein monopole.

Next we turn to the spectrum of massless fermions in this world-volume theory. These typically arise from broken supersymmetry generators. Since type IIB string theory on K3 has 16 unbroken supersymmetries[6] of which 8 are broken in the presence of the Taub-NUT space, we have 8 fermionic zero modes. Since the supersymmetry generators in type IIB on K3 are chiral, the fermionic zero modes associated with broken supersymmetries are also chiral, and are right-moving on the world-sheet. On the other hand if we take type IIB on T^4 we have altogether 32 unbroken supercharges of which 16 are broken in the presence of the Taub-NUT space. Since type IIB on T^4 is a non-chiral theory, we have 8 right-moving and 8 left-moving zero modes.

To summarize, the world-sheet theory describing the dynamics of the Kaluza-Klein monopole always contains 8 bosonic and 8 fermionic right-moving modes. For $\mathcal{M} = K3$ the world-sheet theory has 24 left-moving bosonic modes and no left-moving fermionic modes whereas for $\mathcal{M} = T^4$ the world-sheet theory has 8 left-moving bosonic and 8 left-moving fermionic modes.

We shall now determine the $\widetilde{g}$ transformation properties of these modes. Since $\widetilde{g}$ commutes with the supersymmetries of type IIB on $K3$, all the right-handed fermions living on the world-sheet theory, associated with the broken supersymmetry generators in the presence of Kaluza-Klein monopole, must be neutral under $\widetilde{g}$. Since $\widetilde{g}$ also commutes with the unbroken supersymmetry generators which transforms the right-moving world-

[6] In this section we shall refer to unbroken supersymmetries in various context. Some time it may refer to the symmetry of a given compactification, and some time it will refer to the symmetry of a given brane configuration. The reader must carefully examine the context in which the symmetry is being discussed, since the number of unbroken generators and their action on various fields depend crucially on this information.

sheet fermions into right-moving world-sheet scalars and vice versa, the 8 right-moving bosons on the world-volume of the Kaluza-Klein monopole must also be invariant under $\widetilde{g}$. Five of the left-moving bosons, associated with the 3 transverse degree of freedom and the modes of the 2-form fields along the Taub-NUT space are also invariant under $\widetilde{g}$ since $\widetilde{g}$ acts trivially on the Taub-NUT space. The action of $\widetilde{g}$ on the other P left-moving bosons is represented by its action on the P left-handed 2-forms on $\mathcal{M}$. This completely determines the action of $\widetilde{g}$ on all the $P+5$ left-moving bosons. Since from the analysis of section 2 we know that $\widetilde{g}$ leaves invariant the harmonic 0-form, 4-form and all the three right-handed 2-forms on $\mathcal{M}$, we see that the net action of $\widetilde{g}$ on the $(P+5)$ left-handed bosonic fields on the world-sheet of the Kaluza-Klein monopole is in one to one correspondence with the action of $\widetilde{g}$ on the $(P+5)$ even degree harmonic forms on $\mathcal{M}$, consisting of P left-handed 2-forms, three $\widetilde{g}$ invariant right-handed 2-forms, a $\widetilde{g}$ invariant 0-form and a $\widetilde{g}$ invariant 4-form.

What remains is to determine the action of $\widetilde{g}$ on the left-moving fermions. We shall now show that this can be represented by the action of $\widetilde{g}$ on the harmonic 1- and 3-forms of $\mathcal{M}$. For $\mathcal{M} = K3$ there are no 1- or 3-forms and no left-moving fermions on the world-sheet of the Kaluza-Klein monopole. Hence the result holds trivially. For $\mathcal{M} = T^4$ there are eight left-moving fermions and eight right-moving fermions. These are associated with the sixteen supersymmetry generators which are broken in the presence of a Kaluza-Klein monopole in type IIB string theory on $T^4 \times S^1 \times \widetilde{S}^1$, and hence transform in the spinor representation of the tangent space $\mathrm{SO}(4)_{\|}$ group associated with the T^4 direction. Now $\widetilde{g}$ is an element of this group describing $2\pi/N$ rotation in one plane and $-2\pi/N$ rotation in an orthogonal plane. Translating this into the spinor representation we see that the net effect is to leave half of the eight fermions invariant, rotate two pairs of fermions by $2\pi/N$ and rotate the other two pairs of fermions by $-2\pi/N$. Since we have already seen that the right-moving fermions are neutral under $\widetilde{g}$, the action of $\widetilde{g}$ on the left-moving fermions is to rotate two pairs of fermions by $2\pi/N$ and another two pairs of fermions by $-2\pi/N$. This is identical to the action of $\widetilde{g}$ on the harmonic 1- and 3-forms of T^4 given in (2.10) and (2.12). Thus the problem of studying the $\widetilde{g}$ transformation properties of the left-moving degrees of freedom on the world-sheet reduces to the problem of finding the action of $\widetilde{g}$ on the even and odd degree harmonic forms of $\mathcal{M}$. We now map this problem into an equivalent problem as follows. Let us consider a (4,4) superconformal σ-model in (1+1) dimension with target space $\mathcal{M}$ as described in section 2 and consider the quantity

$$Q_{0,s} = Tr_{\mathrm{RR}}\left((-1)^{F_L+F_R}\widetilde{g}^s e^{2\pi i\tau L_0} e^{-2\pi i\bar{\tau}\bar{L}_0}\right) , \tag{4.9}$$

with $Q_{r,s}$ defined through (2.16). As discussed at the end of section 2, $Q_{0,s}$ counts the difference between the number of even degree harmonic forms and odd degree harmonic forms, weighted by $\widetilde{g}^s$. Using the results of our previous analysis this can be rewritten as

$$\begin{aligned} Q_{0,s} = &\ \text{number of left handed bosons weighted by } \widetilde{g}^s \\ &-\text{number of left handed fermions weighted by } \widetilde{g}^s . \end{aligned} \tag{4.10}$$

Let n_l be the number of left-handed bosons minus fermions carrying $\widetilde{g}$ quantum number

$e^{2\pi il/N}$. Then we have from (4.10)

$$n_l = \frac{1}{N}\sum_{s=0}^{N-1} e^{-2\pi ils/N}\, Q_{0,s}\,. \tag{4.11}$$

Clearly n_l is invariant under $l \to l+N$.

We now turn to the problem of counting the spectrum of BPS excitations of the Kaluza-Klein monopole. First of all note that since there are eight right-moving fermions neutral under $\widetilde{g}$, the zero modes of these fermions are $\mathbb{Z}_N$ invariant. These eight fermionic zero modes may be regarded as the goldstone modes associated with broken supersymmetry generators. Since type IIB string theory on $\mathcal{M}\times S^1/\mathbb{Z}_N$ has 16 supersymmetries, and since a Kaluza-Klein monopole breaks half of these supersymmetries, we expect precisely eight fermionic zero modes associated with the broken supersymmetry generators. Upon quantization this produces a 16-fold degeneracy of states with equal number of bosonic and fermionic states. This is the correct degeneracy of a single irreducible short multiplet representing half BPS states in type IIB string theory compactified on $\mathcal{M}\times S^1/\mathbb{Z}_N$, and will eventually become part of the 64-fold degeneracy of a 1/4 BPS supermultiplet once we tensor this state with the state of the D1-D5 system. Since supersymmetry acts on the right-moving sector of the world-volume theory, BPS condition requires that all the non-zero mode right-moving oscillators are in their ground state. Thus the spectrum of BPS states is obtained by taking the tensor product of this irreducible 16 dimensional supermultiplet with either fermionic or bosonic excitations involving the left-moving degrees of freedom on the world-volume of the Kaluza-Klein monopole. We shall denote by $d_{\rm KK}(l_0')/16$ the degeneracy of states associated with left-moving oscillator excitations carrying total momentum $-l_0'$, weighted by $(-1)^{F_L}$ where F_L denotes the contribution to the space-time fermion number from the left-moving modes on the world-sheet. Thus $d_{\rm KK}(l_0')$ calculates the total degeneracy of half-BPS states weighted by $(-1)^{F_L}$. In order to calculate $d_{\rm KK}(l_0')$ we need to count the number of ways the total momentum $-l_0'$ can be distributed among the different left-moving oscillator excitations, subject to the requirement of $\mathbb{Z}_N$ invariance. Since a mode carrying momentum $-l$ along S^1 picks up a phase of $e^{-2\pi il/N}$ under $2\pi/N$ translation along S^1, it must pick up a phase of $e^{2\pi il/N}$ under $\widetilde{g}$. Thus the number of left-handed bosonic minus fermionic modes carrying momentum l along S^1 is given by n_l given in eq. (4.11). The number $d_{\rm KK}(l_0')/16$ can now be identified as the number of different ways the total momentum l_0' can be distributed among different oscillators, there being n_l bosonic minus fermionic oscillators carrying momentum l. This gives

$$\sum_{l_0'} d_{\rm KK}(l_0')e^{2\pi i\widetilde{\rho}l_0'} = 16\, e^{2\pi iC\widetilde{\rho}} \prod_{l=1}^{\infty}(1-e^{2\pi il\widetilde{\rho}})^{-n_l}\,. \tag{4.12}$$

The constant C represents the l_0' quantum number of the vacuum of the Kaluza-Klein monopole when all oscillators are in their ground state. In order to determine C let us consider the dual asymmetric orbifold description of the system where the Kaluza-Klein monopole gets mapped to an elementary heterotic or type IIA string along S^1. If $\widehat{g}$ denotes the image of g in the asymmetric orbifold description, then since $\widehat{g}$ involves a translation by $2\pi/N$ along S^1, the elementary string along S^1 is in the sector twisted by $\widehat{g}$. Since the modes of the Kaluza-Klein monopole get mapped to the degrees of freedom of the fundamental heterotic or type IIA string, there are n_l left moving bosonic minus fermionic modes which pick up a phase of $e^{2\pi i l/N}$ under the action of $\widehat{g}$. C now represents $N\times$ the contribution to the ground state L_0 eigenvalue from all the left-moving oscillators, — the multiplicative factor of N arising due to the fact that in the orbifold theory the S^1 direction has period $2\pi/N$, and hence the world-sheet σ coordinate of the dual fundamental string is to be identified with $N\times$ the coordinate along S^1. Since a bosonic and a fermionic mode twisted by a phase of $e^{2\pi i\varphi}$ for $0 \le \varphi \le 1$ gives a contribution of $-\frac{1}{24} + \frac{1}{4}\varphi(1-\varphi)$ and $\frac{1}{24} - \frac{1}{4}\varphi(1-\varphi)$ respectively to the ground state L_0 eigenvalue, we have[7]

$$C = -\frac{N}{24}\sum_{l=0}^{N-1} n_l + \frac{N}{4}\sum_{l=0}^{N-1} n_l \,\frac{l}{N}\left(1-\frac{l}{N}\right). \tag{4.13}$$

Using the expression for n_l given in (4.11) we get

$$C = -\frac{1}{24}\sum_{s=0}^{N-1} Q_{0,s}\sum_{l=0}^{N-1} e^{-2\pi i l s/N} + \frac{1}{4}\sum_{s=0}^{N-1} Q_{0,s}\sum_{l=0}^{N-1}\frac{l}{N}\left(1-\frac{l}{N}\right)e^{-2\pi i l s/N}. \tag{4.14}$$

The sum over l can be performed separately for $s=0$ and $s\neq 0$, and yields the answer

$$C = -\widetilde{\alpha}\,, \tag{4.15}$$

with $\widetilde{\alpha}$ defined as in (3.18):

$$\widetilde{\alpha} = \frac{1}{24N}\,Q_{0,0} - \frac{1}{2N}\sum_{s=1}^{N-1} Q_{0,s}\,\frac{e^{-2\pi i s/N}}{(1-e^{-2\pi i s/N})^2}. \tag{4.16}$$

The left-right level matching condition of the dual heterotic string theory guarantees that C and hence $\widetilde{\alpha}$ must be an integer. Using (4.11), (2.18), (4.15) we can rewrite (4.12) as

$$\sum_{l_0'} d_{\rm KK}(l_0')e^{2\pi i\widetilde{\rho} l_0'} = 16\, e^{-2\pi i\widetilde{\alpha}\widetilde{\rho}}\prod_{l=1}^{\infty}(1-e^{2\pi i l\widetilde{\rho}})^{-\sum_{s=0}^{N-1} e^{-2\pi i l s/N}\left(c_0^{(0,s)}(0)+2c_1^{(0,s)}(-1)\right)}. \tag{4.17}$$

4.2 Counting states associated with the overall motion of the D1-D5 system

We shall now turn to the computation of the contribution to the partition function from the overall motion of the D1-D5 system. This has two components, — the center of mass motion of the D1-D5 system along the Taub-NUT space transverse to the plane of the D5-brane, and the dynamics of the Wilson lines on the D5-brane along $\mathcal{M}$. The first component is present irrespective of the choice of $\mathcal{M}$ but the second component exits only if $\mathcal{M}$ has non-contractible one cycles, i.e. for $\mathcal{M} = T^4$.

The contribution from the first component is clearly independent of the choice of $\mathcal{M}$ and has been found in [9]. If $d_{\rm transverse}(l_0, j_0)$ denotes the number of states associated with

[7] We are counting the contribution from a mode and its complex conjugate separately.

the transverse motion of the system, carrying momentum $-l_0$ along S^1 and j_0 along $\widetilde{S}^1$, then we have

$$\sum_{l_0,j_0} d_{\rm transverse}(l_0,j_0)e^{2\pi i l_0\widetilde{\rho}+2\pi i j_0\widetilde{v}} = 4\, e^{-2\pi i\widetilde{v}}\,(1-e^{-2\pi i\widetilde{v}})^{-2}$$
$$\prod_{n=1}^{\infty}\Big\{(1-e^{2\pi i nN\widetilde{\rho}})^4\,(1-e^{2\pi i nN\widetilde{\rho}+2\pi i\widetilde{v}})^{-2}\,(1-e^{2\pi i nN\widetilde{\rho}-2\pi i\widetilde{v}})^{-2}\Big\}\,. \tag{4.18}$$

The factor of 4 comes from the quantization of the right-moving fermionic zero modes [9]. As before, in the counting of states associated with the left-moving oscillators we include a weight factor of $(-1)^{F_L}$. In expressing the right hand side of (4.18) as a series we always expand the terms inside the product in positive powers of $e^{2\pi i\widetilde{\rho}}$. However for the $e^{-2\pi i\widetilde{v}}(1-e^{-2\pi i\widetilde{v}})^{-2}$ factor we have two choices, — either expand it in powers of $e^{-2\pi i\widetilde{v}}$, or rewrite is as $e^{2\pi i\widetilde{v}}(1-e^{2\pi i\widetilde{v}})^{-2}$ and expand it in powers of $e^{2\pi i\widetilde{v}}$. These two different ways of expanding yield different spectrum, and the correct choice depends on the angle between the circles S^1 and $\widetilde{S}^1$ [9, 34, 35]. As this angle passes through 90° the spectrum changes discontinuously.

Let us now compute the contribution to the partition function from the dynamics of the Wilson lines for $\mathcal{M} = T^4$. For this we can ignore the presence of the Kaluza-Klein monopole and the D1-branes, and consider the dynamics of a D5-brane wrapped on $T^4\times S^1$. Taking the T^4 to have small size we can regard the world-volume theory as (1+1) dimensional. This has eight bosonic modes associated with four Wilson lines and four transverse coordinates, but we shall only be interested in the dynamics of the Wilson lines. Similarly there are eight non-chiral fermionic modes, but four of these, related to the transverse bosonic modes by the unbroken supersymmetry algebra that commutes with $\widetilde{g}$, have already been accounted for in the partition function (4.18). Thus we shall consider only four of the fermionic modes which are superpartners of the four Wilson lines under the unbroken supersymmetry algebra.

Now $\widetilde{g}$ acts as a rotation by $2\pi/N$ on one pair of Wilson lines and as a rotation by $-2\pi/N$ on the other pair. Since the unbroken supersymmetry algebra commutes with $\widetilde{g}$ and furthermore, its action on the D5-brane world-volume is non-chiral, $\widetilde{g}$ must act as rotation by $2\pi/N$ on one pair of fermions and $-2\pi/N$ on the other pair both in the left and the right-moving sector. In order to be $\mathbb{Z}_N$ invariant, the modes which pick a phase of $e^{2\pi i/N}$ under $\widetilde{g}$ must carry momentum along S^1 of the form $Nk-1$ for integer k, whereas modes which pick a phase of $e^{-2\pi i/N}$ under $\widetilde{g}$ must carry momentum along S^1 of the form $Nk+1$ for integer k. As a result, both in the left and the right-moving sector, we have a pair of bosons and a pair of fermions carrying S^1 momentum of the form $Nk+1$, and a pair of bosons and a pair of fermions carrying S^1 momentum of the form $Nk-1$.

Eventually when we place this in the background of the Kaluza-Klein monopole, only the supersymmetry associated with the right-moving modes remain unbroken. Thus in order to get a BPS state of the final supersymmetry algebra we must put all the right-moving oscillators in their ground state and consider only left moving excitations.

In order to calculate the partition function associated with these modes we also need information about their j_0 quantum numbers. Near the center of Taub-NUT the j_0 quan-

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tum number corresponds to the sum of the angular momenta in the two planes transverse to the D5-brane. The left- and right-moving bosonic modes associated with the Wilson lines are neutral under rotation in planes transverse to the D5-brane and hence do not carry any j_0 charge. However the fermions, being in the spinor representation of the tangent space group of the transverse space, do carry j_0 charge. Since the net j_0 quantum number is given by the sum of $\pm\frac{1}{2}$ units of angular momentum associated with the two transverse planes, a quarter of the fermions carry $j_0 = 1$, another quarter of them carry $j_0 = -1$ and half of them have $j_0 = 0$. In order to determine which fermions carry $j_0 = \pm 1$ we note that from the point of view of an asymptotic observer j_0 represents momentum along $\widetilde{S}^1$, and hence must commute with the final unbroken supersymmetry generators. Since these generators relate the right-moving bosons with $j_0 = 0$ to the right-moving fermions, the right-moving fermionic excitations must have $j_0 = 0$. Thus the left-moving fermions must have $j_0 = \pm 1$ and hence can be rotated to each other by an appropriate element of the tangent space $\mathrm{SO}(4)_\perp$ group transverse to the D5-brane. Since rotation along the tangent plane transverse to the D5-brane commutes with $\widetilde{g}$, the two left-moving fermions carrying $\widetilde{g}$ quantum number $e^{2\pi i/N}$ must have $j_0 = \pm 1$ and the two left-moving fermions carrying $\widetilde{g}$ quantum number $e^{-2\pi i/N}$ must have $j_0 = \pm 1$.

To summarize, the left-moving excitations on the D5-brane world-volume, related by supersymmetry transformation to the Wilson lines along T^4, consist of four bosonic and four fermionic modes. Two of the four bosonic modes carry momentum along S^1 of the form $Nk+1$ and the other two carry momentum along S^1 of the form $Nk-1$, but neither of them carry any momentum along $\widetilde{S}^1$. On the other hand two of the fermionic modes carry momentum along S^1 of the form $Nk+1$ and ± 1 unit of momentum along $\widetilde{S}^1$, and the other two fermionic modes carry momentum along S^1 of the form $Nk-1$ and ± 1 unit of momentum along $\widetilde{S}^1$. If $d_{\mathrm{wilson}}(l_0, j_0)$ denotes the number of states associated with these modes carrying total momentum $-l_0$ along S^1 and total momentum j_0 along $\widetilde{S}^1$, then

$$\sum_{l_0,j_0} d_{\mathrm{wilson}}(l_0,j_0) e^{2\pi i l_0\widetilde{\rho}+2\pi i j_0\widetilde{v}} = \prod_{\substack{l\in N\mathbb{Z}+1\\ l>0}} (1-e^{2\pi i l\widetilde{\rho}})^{-2} \prod_{\substack{l\in N\mathbb{Z}-1\\ l>0}} (1-e^{2\pi i l\widetilde{\rho}})^{-2} \prod_{\substack{l\in N\mathbb{Z}+1\\ l>0}} (1-e^{2\pi i l\widetilde{\rho}+2\pi i\widetilde{v}})$$
$$\prod_{\substack{l\in N\mathbb{Z}+1\\ l>0}} (1-e^{2\pi i l\widetilde{\rho}-2\pi i\widetilde{v}}) \prod_{\substack{l\in N\mathbb{Z}-1\\ l>0}} (1-e^{2\pi i l\widetilde{\rho}+2\pi i\widetilde{v}}) \prod_{\substack{l\in N\mathbb{Z}-1\\ l>0}} (1-e^{2\pi i l\widetilde{\rho}-2\pi i\widetilde{v}}) \tag{4.19}$$

Using (2.9), (2.14) one can show that the partition function associated with the overall dynamics of the D1-D5 system, given by the product of the contribution (4.18) from the dynamics of the transverse modes and (in case $\mathcal{M} = T^4$) the contribution (4.19) from the dynamics of the Wilson lines along T^4, can be written as

$$\sum_{l_0,j_0} d_{\mathrm{CM}}(l_0,j_0) e^{2\pi i l_0\widetilde{\rho}+2\pi i j_0\widetilde{v}} = 4\, e^{-2\pi i\widetilde{v}} \prod_{l=1}^{\infty} (1-e^{2\pi i l\widetilde{\rho}})^{2\sum_{s=0}^{N-1} e^{-2\pi i l s/N} c_1^{(0,s)}(-1)}$$
$$\prod_{l=1}^{\infty} (1-e^{2\pi i l\widetilde{\rho}+2\pi i\widetilde{v}})^{-\sum_{s=0}^{N-1} e^{-2\pi i l s/N} c_1^{(0,s)}(-1)} \prod_{l=0}^{\infty} (1-e^{2\pi i l\widetilde{\rho}-2\pi i\widetilde{v}})^{-\sum_{s=0}^{N-1} e^{-2\pi i l s/N} c_1^{(0,s)}(-1)} \tag{4.20}$$

both for $\mathcal{M} = K3$ and $\mathcal{M} = T^4$.

4.3 Counting states associated with the relative motion of the D1-D5 system

Finally we turn to the problem of counting states associated with the motion of the D1-brane in the plane of the D5-brane. This can be done by following a procedure identical to the one described in [9] (which in turn is a generalization of the analysis of [36]) and yields the answer:

$$\sum_{Q_1,L,J'} d_{D1}(Q_1,L,J')e^{2\pi i(\widetilde{\sigma}Q_1/N+\widetilde{\rho}L+\widetilde{v}J')} = \prod_{\substack{w,l,j\in\mathbb{Z}\\ w>0,l\geq 0}} \left(1-e^{2\pi i(\widetilde{\sigma}w/N+\widetilde{\rho}l+\widetilde{v}j)}\right)^{-n(w,l,j)}, \quad (4.21)$$

where

$$n(w,l,j) = \sum_{s=0}^{N-1} e^{-2\pi isl/N} c_b^{(r,s)}(4lw/N-j^2)\,, \qquad r = w \bmod N,\; b = j \bmod 2\,. \quad (4.22)$$

4.4 The full partition function

Using (4.8), (4.17), (4.20) and (4.21) we now get

$$f(\widetilde{\rho},\widetilde{\sigma},\widetilde{v}) = e^{-2\pi i(\widetilde{\alpha}\widetilde{\rho}+\widetilde{v})} \prod_{b=0}^{1}\prod_{r=0}^{N-1} \prod_{\substack{k'\in\mathbb{Z}+\frac{r}{N},l\in\mathbb{Z},j\in2\mathbb{Z}+b\\ k',l\geq 0, j<0 \text{ for } k'=l=0}} (1-e^{2\pi i(\widetilde{\sigma}k'+\widetilde{\rho}l+\widetilde{v}j)})^{-\sum_{s=0}^{N-1} e^{-2\pi isl/N} c_b^{(r,s)}(4lk'-j^2)}\,. \quad (4.23)$$

The $k'=0$ term in the last expression comes from the terms involving $d_{\rm CM}(l_0,j_0)$ and $d_{\rm KK}(l'_0)$. Comparing the right hand side of this equation with the expression for $\widetilde{\Phi}$ given in (3.16) we can rewrite (4.8) as

$$f(\widetilde{\rho},\widetilde{\sigma},\widetilde{v}) = \frac{e^{2\pi i\widetilde{\gamma}\widetilde{\sigma}}}{\widetilde{\Phi}(\widetilde{\rho},\widetilde{\sigma},\widetilde{v})}\,, \quad (4.24)$$

where, from (3.18),

$$\widetilde{\gamma}\,N = \frac{1}{24}\,Q_{0,0} = \frac{1}{24}\chi(\mathcal{M})\,. \quad (4.25)$$

Eq. (4.7) now gives

$$h(Q_1,n,J) = \frac{1}{N}\int_{\mathcal{C}} d\widetilde{\rho}\,d\widetilde{\sigma}\,d\widetilde{v}\,e^{-2\pi i(\widetilde{\rho}n+\widetilde{\sigma}(Q_1-\widetilde{\gamma}\,N)/N+\widetilde{v}J)}\,\frac{1}{\widetilde{\Phi}(\widetilde{\rho},\widetilde{\sigma},\widetilde{v})}\,, \quad (4.26)$$

where $\mathcal{C}$ is a three real dimensional subspace of the three complex dimensional space labelled by $(\widetilde{\rho},\widetilde{\sigma},\widetilde{v})$, given by

$$\begin{aligned} &Im\,\widetilde{\rho} = M_1, \quad Im\,\widetilde{\sigma} = M_2, \quad Im\,\widetilde{v} = M_3,\\ &0\leq Re\,\widetilde{\rho}\leq 1, \quad 0\leq Re\,\widetilde{\sigma}\leq N, \quad 0\leq Re\,\widetilde{v}\leq 1\,. \end{aligned} \quad (4.27)$$

M_1, M_2 and M_3 are large but fixed positive numbers. Identifying $h(Q_1,n,J)$ with the degeneracy $d(Q_e,Q_m)$, using (4.2), and noting that β defined in (4.1) is equal to $\widetilde{\gamma}N$ given in (4.25), we can rewrite (4.26) as

$$d(Q_e,Q_m) = \frac{1}{N}\int_{\mathcal{C}} d\widetilde{\rho}\,d\widetilde{\sigma}\,d\widetilde{v}\,e^{-\pi i(N\widetilde{\rho}Q_e^2+\widetilde{\sigma}Q_m^2/N+2\widetilde{v}Q_e\cdot Q_m)}\,\frac{1}{\widetilde{\Phi}(\widetilde{\rho},\widetilde{\sigma},\widetilde{v})}\,. \quad (4.28)$$

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5. S-duality invariance of $d(Q_e, Q_m)$

The proof of S-duality invariance of $d(Q_e, Q_m)$ proceeds as in [6, 10]. As described in (4.4), under the action of S-duality the electric and magnetic charges transform to

$$
\begin{aligned}
Q_e \to Q_e' &= a\,Q_e + b\,Q_m, \\
Q_m \to Q_m' &= c\,Q_e + d\,Q_m\,, \\
ad - bc = 1, &\quad a, d \in N\mathbb{Z}+1,\ b \in \mathbb{Z},\ c \in N\mathbb{Z}\,.
\end{aligned} \tag{5.1}
$$

Let us define

$$
\begin{aligned}
\widetilde{\Omega} &\equiv \begin{pmatrix} \widetilde{\rho} & \widetilde{v} \\ \widetilde{v} & \widetilde{\sigma} \end{pmatrix}, \\
\widetilde{\Omega}' &\equiv \begin{pmatrix} \widetilde{\rho}' & \widetilde{v}' \\ \widetilde{v}' & \widetilde{\sigma}' \end{pmatrix} \\
&= (\widetilde{A}\widetilde{\Omega} + \widetilde{B})(\widetilde{C}\widetilde{\Omega} + \widetilde{D})^{-1}, \\
\begin{pmatrix} \widetilde{A} & \widetilde{B} \\ \widetilde{C} & \widetilde{D} \end{pmatrix} &= \begin{pmatrix} \tilde{a} & -\tilde{b} & 0 & 0 \\ -\tilde{c} & \tilde{d} & 0 & 0 \\ 0 & 0 & \tilde{d} & \tilde{c} \\ 0 & 0 & \tilde{b} & \tilde{a} \end{pmatrix}
\end{aligned} \tag{5.2}
$$

where

$$
\begin{pmatrix} \tilde{a} & \tilde{b} \\ \tilde{c} & \tilde{d} \end{pmatrix} = \begin{pmatrix} d & c/N \\ bN & a \end{pmatrix}. \tag{5.3}
$$

This gives

$$
\begin{aligned}
\widetilde{\rho}' &= \tilde{a}^2\widetilde{\rho} + \tilde{b}^2\widetilde{\sigma} - 2\tilde{a}\tilde{b}\widetilde{v}\,, \\
\widetilde{\sigma}' &= \tilde{c}^2\widetilde{\rho} + \tilde{d}^2\widetilde{\sigma} - 2\tilde{c}\tilde{d}\widetilde{v}\,, \\
\widetilde{v}' &= -\tilde{a}\tilde{c}\widetilde{\rho} - \tilde{b}\tilde{d}\widetilde{\sigma} + (\tilde{a}\tilde{d} + \tilde{b}\tilde{c})\widetilde{v}\,.
\end{aligned} \tag{5.4}
$$

Using (5.1), (5.3), (5.4) and the quantization laws of Q_e^2, Q_m^2 and $Q_e \cdot Q_m$ one can easily verify that

$$
e^{-\pi i(N\widetilde{\rho}Q_e^2 + \widetilde{\sigma}Q_m^2/N + 2\widetilde{v}Q_e\cdot Q_m)} = e^{-\pi i(N\widetilde{\rho}'Q_e'^2 + \widetilde{\sigma}'Q_m'^2/N + 2\widetilde{v}'Q_e'\cdot Q_m')}\,, \tag{5.5}
$$

and

$$
d\widetilde{\rho}\,d\widetilde{\sigma}\,d\widetilde{v} = d\widetilde{\rho}'\,d\widetilde{\sigma}'\,d\widetilde{v}'\,. \tag{5.6}
$$

One can also verify that the transformation described in (5.2) is an element of the group $\widetilde{G}$ under which $\widetilde{\Phi}$ transforms as a modular form of weight k.[8] Since for the transformation (5.2), $\det(\widetilde{C}\widetilde{\Omega} + \widetilde{D}) = 1$, we have

$$
\widetilde{\Phi}(\widetilde{\rho}', \widetilde{\sigma}', \widetilde{v}') = \widetilde{\Phi}(\widetilde{\rho}, \widetilde{\sigma}, \widetilde{v})\,. \tag{5.7}
$$

[8] This can be seen directly from the fact that under the transformation (5.2) the integral $\widetilde{\mathcal{I}}_{r,s,b}$ given in (3.4) remains unchanged after a suitable relabelling of the indices m_1, n_1, m_2, n_2. Eq. (3.13) then tells us that $\widetilde{\Phi}(\widetilde{\rho}, \widetilde{\sigma}, \widetilde{v})$ transforms as a modular form of weight k under this transformation.

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Finally we note that under the map (5.4) the three cycle $\mathcal{C}$ gets mapped to itself up to a shift that can be removed with the help of the shift symmetries

$$\widetilde{\widetilde{\Phi}}(\widetilde{\rho},\widetilde{\sigma},\widetilde{v}) = \widetilde{\widetilde{\Phi}}(\widetilde{\rho}+1,\widetilde{\sigma},\widetilde{v}) = \widetilde{\widetilde{\Phi}}(\widetilde{\rho},\widetilde{\sigma}+N,\widetilde{v}) = \widetilde{\widetilde{\Phi}}(\widetilde{\rho},\widetilde{\sigma},\widetilde{v}+1)\,, \tag{5.8}$$

which are manifest from (3.16). Thus eqs. (5.5)- (5.7) allow us to express (4.28) as

$$d(Q_e,Q_m) = \frac{1}{N}\int_{\mathcal{C}} d\widetilde{\rho}'\, d\widetilde{\sigma}'\, d\widetilde{v}'\, e^{-\pi i(N\widetilde{\rho}' Q_e'^2 + \widetilde{\sigma}' Q_m'^2/N + 2\widetilde{v} Q_e'\cdot Q_m')}\,\frac{1}{\widetilde{\widetilde{\Phi}}(\widetilde{\rho}',\widetilde{\sigma}',\widetilde{v}')} = d(Q_e',Q_m')\,. \tag{5.9}$$

This proves invariance of $d(Q_e,Q_m)$ under the S-duality group $\Gamma_1(N)$.

6. Statistical entropy function

In this section we shall describe the behaviour of $d(Q_e,Q_m)$ for large charges, and also compute the first order corrections to the leading asymptotic formula. Our starting point is the expression (4.28) for $d(Q_e,Q_m)$:

$$d(Q_e,Q_m) = \frac{1}{N}\int_{\mathcal{C}} d\widetilde{\rho} d\widetilde{\sigma} d\widetilde{v} \frac{1}{\widetilde{\widetilde{\Phi}}(\widetilde{\rho},\widetilde{\sigma},\widetilde{v})} \exp\left[-i\pi(N\widetilde{\rho} Q_e^2 + \widetilde{\sigma} Q_m^2/N + 2\widetilde{v} Q_e\cdot Q_m)\right]. \tag{6.1}$$

Using eqs. (3.7), (3.19) and the result

$$d\widetilde{\rho} d\widetilde{\sigma} d\widetilde{v} = (2v-\rho-\sigma)^{-3} d\rho d\sigma dv\,, \tag{6.2}$$

we can rewrite (6.1) as

$$\begin{aligned} d(Q_e,Q_m) &= \frac{1}{N\,C_1}\int_{\mathcal{C}'} d\rho d\sigma dv\,(2v-\rho-\sigma)^{-k-3}\,\frac{1}{\widetilde{\Phi}(\rho,\sigma,v)} \\ &\quad \exp\left[-i\pi\left\{\frac{v^2-\rho\sigma}{2v-\rho-\sigma}Q_m^2 + \frac{1}{2v-\rho-\sigma}Q_e^2 + \frac{2(v-\rho)}{2v-\rho-\sigma}Q_e\cdot Q_m\right\}\right] \end{aligned} \tag{6.3}$$

where $\mathcal{C}'$ is the image of $\mathcal{C}$ under the map (3.8). We can evaluate this integral by first performing the v integral using Cauchy's formula and then carrying out the ρ and σ integrals by saddle point approximation. Following the analysis of [1, 9] we can show that the dominant contribution comes from the pole at

$$\widetilde{\rho}\widetilde{\sigma} - \widetilde{v}^2 + \widetilde{v} = 0 \quad i.e. \quad v = 0\,. \tag{6.4}$$

From (3.20) we see that contribution from this pole is given by

$$\begin{aligned} d(Q_e,Q_m) &\simeq C_0 \int_{\mathcal{C}''} d\rho d\sigma dv\, v^{-2}\,(2v-\rho-\sigma)^{-k-3}\,(g(\rho)g(\sigma))^{-1} \\ &\quad \exp\left[-i\pi\left\{\frac{v^2-\rho\sigma}{2v-\rho-\sigma}Q_m^2 + \frac{1}{2v-\rho-\sigma}Q_e^2 + \frac{2(v-\rho)}{2v-\rho-\sigma}Q_e\cdot Q_m\right\}\right], \end{aligned} \tag{6.5}$$

where $\mathcal{C}''$ is a contour around $v = 0$, C_0 is a constant and $g(\rho)$ has been defined in (3.21). This integral is exactly of the form given in eq. (4.19) of [9]. Thus subsequent analysis of this integral can be done following the procedure of [9], and we arrive at the result that the statistical entropy

$$S_{\rm stat}(Q_e, Q_m) \equiv \ln d(Q_e, Q_m) \tag{6.6}$$

is obtained by extremizing the statistical entropy function

$$-\widetilde{\Gamma}_B(\vec{\tau}) = \frac{\pi}{2\tau_2} |Q_e + \tau Q_m|^2 - \ln g(\tau) - \ln g(-\bar{\tau}) - (k+2)\ln(2\tau_2) + \text{constant} + \mathcal{O}(Q^{-2}) \tag{6.7}$$

with respect to the real and imaginary parts of τ.

7. Black hole entropy function

We now turn to the computation of the entropy of a black hole carrying charge quantum numbers (Q_e, Q_m) and compare it with the statistical entropy computed in section 6. For this we first need to determine the effective action governing the dynamics of the theory. The leading order entropy is determined by the low energy effective action with two derivative terms. This is the standard action of $\mathcal{N} = 4$ supergravity theories. An important class of four derivative corrections to the action is the Gauss-Bonnet term. We shall now turn to the computation of this term. The calculation is best carried out in the original description of the theory as type IIB string theory compactified on $(\mathcal{M} \times S^1 \times \widetilde{S}^1)/\mathbb{Z}_N$. We shall denote by $t = t_1 + it_2$ and $u = u_1 + iu_2$ the Kahler and complex structure moduli of the torus $S^1 \times \widetilde{S}^1$ with the normalization convention that is appropriate for the orbifold theory. Thus for example if R_1 and R_2 denote the radii of $\widetilde{S}^1$ and S^1 measured in the string metric, and if the off-diagonal components of the metric and the anti-symmetric tensor field are zero, then we shall take $t_2 = R_1 R_2/N$ and $u_2 = R_2/(R_1 N)$, taking into account the fact that in the orbifold theory the various fields have $\widetilde{g}$-twisted boundary condition under a $2\pi R_2/N$ translation along S^1 and $2\pi R_1$ translation along $\widetilde{S}^1$. In the same spirit we shall choose the units of momentum along S^1 and $\widetilde{S}^1$ to be N/R_2 and $1/R_1$ respectively, and unit of winding charge along S^1 and $\widetilde{S}^1$ to be $2\pi R_2/N$ and $2\pi R_1$ respectively. Thus for example a one unit of winding charge along S^1 actually represents a twisted sector state, with twist g. It is known that one loop quantum corrections in this theory give rise to a Gauss-Bonnet contribution to the effective Lagrangian density of the form [23]:

$$\Delta\mathcal{L} = \phi(u, \bar{u}) \left\{ R_{G\mu\nu\rho\sigma} R_G^{\mu\nu\rho\sigma} - 4 R_{G\mu\nu} R_G^{\mu\nu} + R_G^2 \right\} , \tag{7.1}$$

where $\phi(u, \bar{u})$ is a function to be determined. Note in particular that ϕ is independent of the Kahler modulus t of $S^1 \times \widetilde{S}^1$. The analysis of [23] shows that $\phi(u, \bar{u})$ is given by the relation:

$$\partial_u \phi(u, \bar{u}) = \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2} \, \partial_u B_4 \, , \tag{7.2}$$

where B_4 is defined as follows. Let us consider type IIB string theory compactified on $(\mathcal{M} \times S^1 \times \widetilde{S}^1)/\mathbb{Z}_N$ in the light-cone gauge Green-Schwarz formulation, denote by Tr^f the trace over all states in this theory excluding the momentum modes associated with the non-compact directions (since their effect has been already included in arriving at the τ_2 factor in (7.2)) and denote by L_0^f, $\bar{L}_0^f$ the Virasoro generators associated with the left and the right-moving modes, excluding the contribution from the momentum modes associated with the non-compact directions. We also define F_L^f, F_R^f to be the contribution to the space-time fermion numbers from the left and the right-moving modes on the world-sheet.

In this case

$$\begin{aligned} B_4 &= K\, Tr^f \left(q^{L_0^f} \bar{q}^{\bar{L}_0^f} (-1)^{F_L^f + F_R^f} h^4 \right), \\ q &\equiv e^{2\pi i \tau} \end{aligned} \tag{7.3}$$

where K is a constant to be determined later and h denotes the total helicity of the state. The evaluation of the right hand side of (7.3) proceeds as follows. We first note that without the h^4 term the answer will vanish due to the fermion zero mode contribution to the trace since quantization of a conjugate pair of fermion zero modes $(\psi_0, \psi_0^\dagger)$ gives rise to a pair of states with opposite $(-1)^{F_L^f + F_R^f}$. This can be avoided if we insert a factor of h in the trace and pick the contribution to h from this particular conjugate pair of fermions since the two states have the same $(-1)^{F_L^f + F_R^f}\, h$ quantum numbers. This can be repeated for every pair of conjugate fermions. In the present example we have altogether 8 fermion zero modes which are neutral under the orbifold group $\mathbb{Z}_N$, – 4 from the left-moving sector and 4 from the right-moving sector. As a result we need four factors of h to soak up all the fermion zero modes. Thus in effect we can simplify (7.3) by expressing it as

$$B_4 = K' Tr^{f\prime} \left(q^{L_0^f} \bar{q}^{\bar{L}_0^f} (-1)^{F_L^f + F_R^f} \right) \tag{7.4}$$

where K' is a different normalization constant and the prime in the trace denotes that we should ignore the effect of fermion zero modes in evaluating the trace. Since we are using the Green-Schwarz formulation, the 4 left-moving and 4 right-moving fermions which are neutral under the orbifold group $\mathbb{Z}_N$ satisfy periodic boundary condition. Thus the effect of the non-zero mode oscillators associated with these fermions cancel against the contribution from the non-zero mode bosonic oscillators associated with the circles S^1 and $\widetilde{S}^1$ and the two non-compact directions. This leads to a further simplification in which the trace can be taken over only the degrees of freedom associated with the compact space $\mathcal{M}$ and the bosonic zero modes associated with the circles S^1 and $\widetilde{S}^1$. The latter includes the quantum numbers m_1 and m_2 denoting the number of units of momentum along $\widetilde{S}^1$ and S^1, and the quantum numbers n_1 and n_2 denoting the number of units of winding along $\widetilde{S}^1$ and S^1. The units of momentum and winding along the two circles are chosen according to the convention described earlier. Thus for example m_2 unit of momentum along S^1 will correspond to a physical momentum of Nm_2/R_2 in string units. This shows that m_2 can be fractional, being quantized in units of $1/N$. On the other hand a sector with n_2 unit of winding along S^1 describes a fundamental string of length $2\pi n_2 R_2/N$, and hence this state belongs to a sector twisted by g^{n_2}.[9] In this convention the contributions to $\bar{L}_0^f$ and

[9] This picture can be called the view from ‘downstairs’. In contrast if we use parameters and units which are natural for the theory before orbifolding, it corresponds to the view from ‘upstairs’.

L_0^f from the bosonic zero modes associated with $S^1 \times \widetilde{S}^1$ are given by, respectively,

$$\begin{aligned} \frac{1}{2}k_R^2 &= \frac{1}{4t_2u_2}|-m_1u+m_2+n_1t+n_2tu|^2\,, \\ \frac{1}{2}k_L^2 &= \frac{1}{2}k_R^2+m_1n_1+m_2n_2\,. \end{aligned} \tag{7.5}$$

Thus (7.4) may now be rewritten as

$$B_4 = \frac{K'}{N}\sum_{r=0}^{N-1}\sum_{s=0}^{N-1}\sum_{\substack{m_1,n_1\in\mathbb{Z},m_2\in\mathbb{Z}/N\\ n_2\in N\mathbb{Z}+r}} q^{k_L^2/2}\bar{q}^{k_R^2/2}e^{2\pi i m_2 s}Tr_{RR,\widetilde{g}^r}\left((-1)^{F_L+F_R}\widetilde{g}^s q^{L_0}\bar{q}^{\bar{L}_0}\right)\,. \tag{7.6}$$

The sum over s in (7.6) arises from the insertion of the projection operator $\frac{1}{N}\sum_{s=0}^{N-1}g^s$ in the trace, while the sum over r represents the sum over various twisted sector states. $Tr_{RR;\widetilde{g}^r}$ denotes trace over the $\widetilde{g}^r$-twisted sector RR states of the (4,4) superconformal field theory with target space $\mathcal{M}$. As required, the quantum number n_2 that determines the part of g-twist along S^1 is correlated with the integer r that determines the amount of g-twist along $\mathcal{M}$. The $e^{2\pi i m_2 s}$ factor represents part of g^s that acts as translation along S^1 while the action of g^s on $\mathcal{M}$ is represented by the operator $\widetilde{g}^s$ inserted into the trace. We now note that the trace part in (7.6) is precisely the quantity $N\,F^{(r,s)}(\tau, z=0)$ defined in (2.2) for $q=e^{2\pi i\tau}$. Thus we can rewrite (7.6) as

$$B_4 = K'\sum_{r=0}^{N-1}\sum_{s=0}^{N-1}\sum_{\substack{m_1,n_1\in\mathbb{Z},m_2\in\mathbb{Z}/N\\ n_2\in N\mathbb{Z}+r}} q^{k_L^2/2}\bar{q}^{k_R^2/2}e^{2\pi i m_2 s}F^{(r,s)}(\tau,0)\,. \tag{7.7}$$

We shall now compare (7.7) with the expression for $\widehat{\mathcal{I}}(\widetilde{\rho},\widetilde{\sigma},\widetilde{v})$ given in (3.3), (3.5) at $\widetilde{\rho}=u$, $\widetilde{\sigma}=t$ and $\widetilde{v}=0$. In this case p_R^2, p_L^2 defined in (3.2) reduces to k_R^2 and $k_L^2+\frac{1}{2}j^2$ respectively, with k_R^2, k_L^2 given in (7.5). As a result we have

$$\begin{aligned} \widehat{\mathcal{I}}(u,t,0) &= \int_{\mathcal{F}}\frac{d^2\tau}{\tau_2}\sum_{r,s=0}^{N-1}\sum_{b=0}^{1}\sum_{\substack{m_1,n_1\in\mathbb{Z},m_2\in\mathbb{Z}/N\\ n_2\in N\mathbb{Z}+r,j\in 2\mathbb{Z}+b}} q^{k_L^2/2}\bar{q}^{k_R^2/2}q^{j^2/4}e^{2\pi i m_2 s}h_b^{(r,s)}(\tau) \\ &= \int_{\mathcal{F}}\frac{d^2\tau}{\tau_2}\sum_{r,s=0}^{N-1}\sum_{\substack{m_1,n_1\in\mathbb{Z},m_2\in\mathbb{Z}/N\\ n_2\in N\mathbb{Z}+r}} q^{k_L^2/2}\bar{q}^{k_R^2/2}e^{2\pi i m_2 s}(\vartheta_3(2\tau,0)h_0(\tau)+\vartheta_2(2\tau,0)h_1(\tau)) \\ &= \int_{\mathcal{F}}\frac{d^2\tau}{\tau_2}\sum_{r,s=0}^{N-1}\sum_{\substack{m_1,n_1\in\mathbb{Z},m_2\in\mathbb{Z}/N\\ n_2\in N\mathbb{Z}+r}} q^{k_L^2/2}\bar{q}^{k_R^2/2}e^{2\pi i m_2 s}F^{(r,s)}(\tau,0)\,, \end{aligned} \tag{7.8}$$

where in the last step we have used eq. (2.3). Comparing (7.7) with (7.8) we see that

$$\int_{\mathcal{F}}\frac{d^2\tau}{\tau_2}B_4 = K'\,\widehat{\mathcal{I}}(u,t,0)\,. \tag{7.9}$$

Using (3.14) and (3.20) we get

$$\int_{\mathcal{F}} \frac{d^2\tau}{\tau_2} B_4 = -2\,K' \lim_{v\to 0} \left(k \ln t_2 + k \ln u_2 + 2\ln v + 2\ln \bar{v} + \ln g(t) + \ln g(\bar{t})\right.$$
$$\left. + \ln g(u) + \ln g(\bar{u})\right) + \text{constant}\,. \tag{7.10}$$

Naively the right hand side diverges in the $v \to 0$ limit. The origin of this infinity lies in the fact that *a priori* the integral $\widehat{\mathcal{I}}$ as well as $\int d^2\tau B_4/\tau_2$ has divergences from integration over the large τ_2 region which needs to be removed by adding constant terms in the integrand. The constants which need to be added to the integrand of $K'\widehat{\mathcal{I}}$ is different from the one that needs to be added to B_4. Once we take into account this difference the right hand side of (7.10) should become finite. In order to achieve this we shall first regularize the right hand side of (7.10) and then remove the divergent part by subtraction. Since the original regularization where we add an additive constant to the integrand is duality invariant, we must regularize the right hand side of (7.10) in a duality invariant manner. Now under a duality transformation of the form $t \to (at+b)/(ct+d)$, v transforms to $v/(ct+d)$. Similarly under a duality transformation of the form $u \to (pu+q)/(ru+s)$, v transforms to $v/(ru+s)$. Thus the combination $v\bar{v}/(t_2 u_2)$ is invariant under both types of duality transformation. This suggests that a natural way to remove the divergence on the right hand side of (7.10) is to replace $v\bar{v}/(t_2u_2)$ by a small constant ϵ and then remove the $\ln\epsilon$ pieces by subtraction. This gives

$$\int_{\mathcal{F}} \frac{d^2\tau}{\tau_2} B_4 = -2\,K' \left((k+2)\ln t_2 + (k+2)\ln u_2 + \ln g(t) + \ln g(\bar{t})\right.$$
$$\left. + \ln g(u) + \ln g(\bar{u})\right) + \text{constant}\,. \tag{7.11}$$

Comparing (7.2) with (7.11) we now get

$$\phi(u,\bar{u}) = -2\,K' \left((k+2)\ln u_2 + \ln g(u) + \ln g(\bar{u})\right) + \text{constant}\,. \tag{7.12}$$

We now turn to the determination of K'. This constant is universal independent of the specific theory we are analysing. Thus we can find it by working with the type IIB string theory compactified on $K3 \times S^1 \times \widetilde{S}^1$. In this case $k = 10$ and $g(\tau) = \eta(\tau)^{24}$. This matches with the known answer [28, 29] for $\phi(u,\bar{u})$ if we choose $K' = 1/(128\pi^2)$. Thus we have

$$\phi(u,\bar{u}) = -\frac{1}{64\pi^2}\left((k+2)\ln u_2 + \ln g(u) + \ln g(\bar{u})\right) + \text{constant}\,. \tag{7.13}$$

Under the duality map that relates type IIB string theory on the $\mathbb{Z}_N$ orbifold of $\mathcal{M}\times S^1 \times \widetilde{S}^1$ to an asymmetric $\mathbb{Z}_N$ orbifold of heterotic or type IIA string theory on T^6, the modulus u of the original type IIB string theory gets related to the axion-dilaton modulus $\tau = a + iS$ of the final asymmetric orbifold theory. Thus in this description the Gauss-Bonnet term in the effective Lagrangian density takes the form

$$\Delta\mathcal{L} = \phi(\tau,\bar{\tau})\,\left\{R_{G\mu\nu\rho\sigma}R_G^{\mu\nu\rho\sigma} - 4R_{G\mu\nu}R_G^{\mu\nu} + R_G^2\right\}\,, \tag{7.14}$$

with

$$\phi(\tau,\bar{\tau}) = -\frac{1}{64\pi^2}\left((k+2)\ln \tau_2 + \ln g(\tau) + \ln g(\bar{\tau})\right) + \text{constant}\,. \tag{7.15}$$

The effect of this term on the computation of the black hole entropy was analyzed in [22]. The resulting entropy function, after elimination of all the near horizon parameters except the axion-dilaton field τ, is

$$\mathcal{E} = \frac{\pi}{2\tau_2}\,|Q_e + \tau Q_m|^2 - \ln g(\tau) - \ln g(-\bar\tau) - (k+2)\ln(2\tau_2) + \text{constant} + \mathcal{O}(Q^{-2})\,. \quad (7.16)$$

The black hole entropy is obtained by extremizing this function with respect to the real and imaginary parts of τ. Since the black hole entropy function coincides with the statistical entropy function given in (6.7), we see that the black hole entropy agrees with the statistical entropy to this order.

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